

# Approximate computation and implicit regularization for very large-scale data analysis

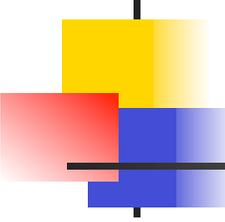
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**Michael W. Mahoney**

Stanford University

May 2012

*(For more info, see: <http://cs.stanford.edu/people/mmahoney>)*



# Algorithmic vs. Statistical Perspectives

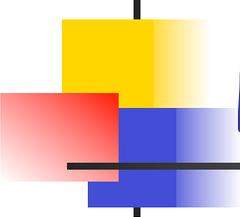
Lambert (2000); Mahoney "Algorithmic and Statistical Perspectives on Large-Scale Data Analysis" (2010)

## Computer Scientists

- *Data*: are a **record of everything** that happened.
- *Goal*: process the data to **find interesting patterns** and associations.
- *Methodology*: Develop approximation algorithms under different models of data access since the goal is typically **computationally hard**.

## Statisticians (and Natural Scientists, etc)

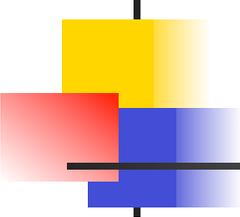
- *Data*: are a **particular random instantiation** of an underlying process describing unobserved patterns in the world.
- *Goal*: is to **extract information** about the world from noisy data.
- *Methodology*: Make inferences (perhaps about unseen events) by **positing a model** that describes the random variability of the data around the deterministic model.



# Perspectives are NOT incompatible

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- Statistical/probabilistic ideas are central to recent work on developing improved **randomized algorithms for matrix problems**.
- Intractable optimization problems on graphs/networks yield to approximation when **assumptions are made about network participants**.
- In boosting (a statistical technique that fits an additive model by minimizing an objective function with a method such as gradient descent), the **computation parameter** (i.e., the number of iterations) also serves as a **regularization parameter**.



## But they are VERY different paradigms

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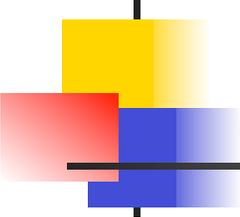
### **Statistics, natural sciences, scientific computing, etc:**

- Problems often involve computation, but the study of *computation per se is secondary*
- Only makes sense to develop algorithms for *well-posed\** problems
- First, write down a model, and think about computation later

### **Computer science:**

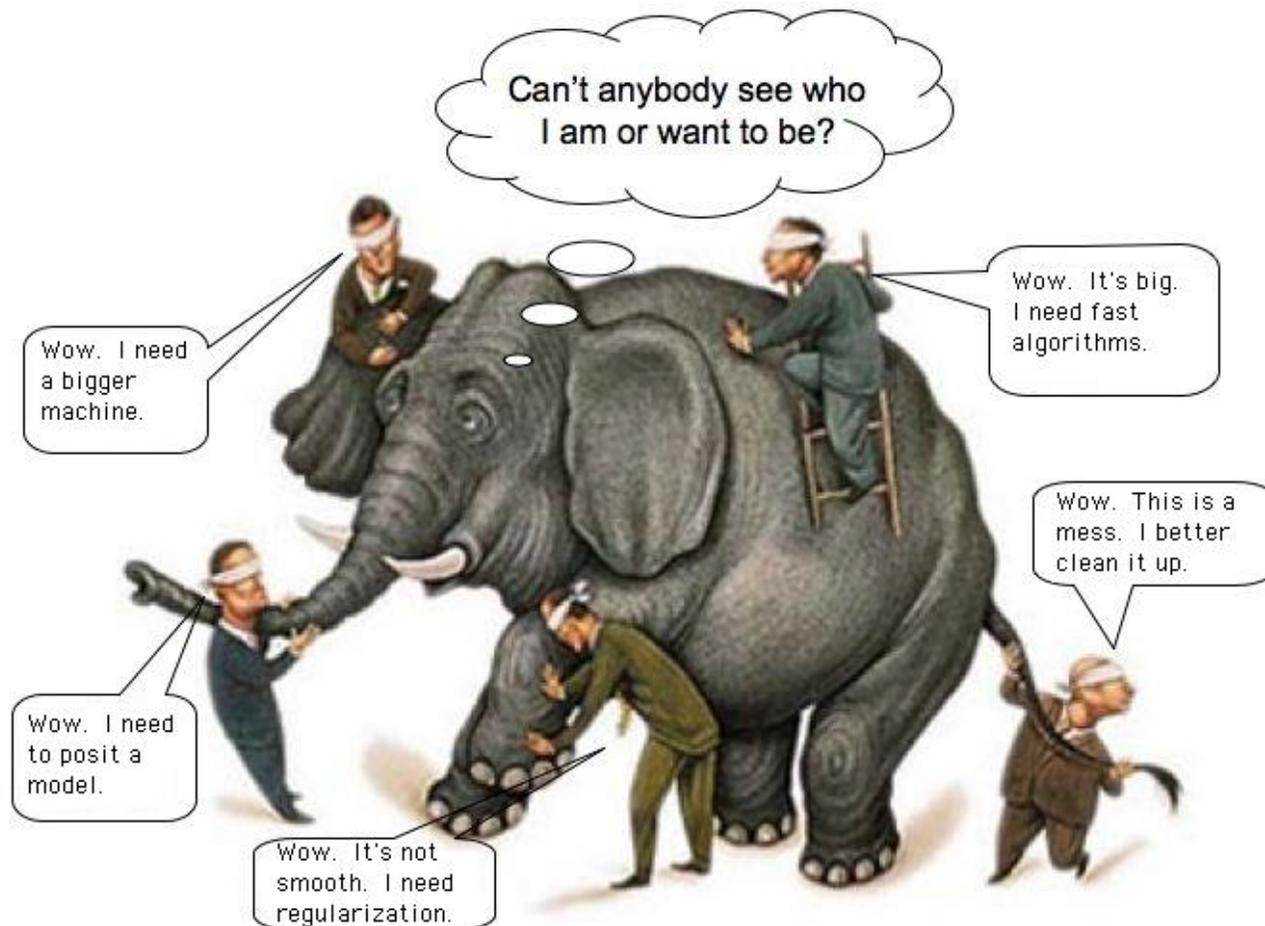
- Easier to study *computation per se in discrete settings*, e.g., Turing machines, logic, complexity classes
- Theory of algorithms *divorces computation from data*
- First, run a fast algorithm, and ask what it means later

\*Solution exists, is unique, and varies continuously with input data



# How do we view BIG data?

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# Anecdote 1: Randomized Matrix Algorithms

Mahoney "Algorithmic and Statistical Perspectives on Large-Scale Data Analysis" (2010)  
Mahoney "Randomized Algorithms for Matrices and Data" (2011)

## Theoretical origins

- theoretical computer science, convex analysis, etc.
- Johnson-Lindenstrauss
- Additive-error algs
- Good worst-case analysis
- No statistical analysis



## Practical applications

- NLA, ML, statistics, data analysis, genetics, etc
- Fast JL transform
- Relative-error algs
- Numerically-stable algs
- Good statistical properties

How to "bridge the gap"?

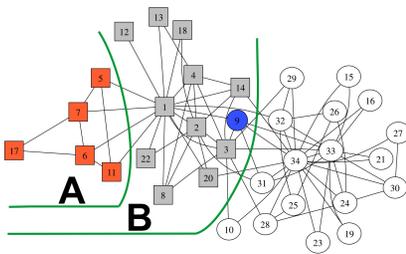
- decouple randomization from linear algebra
- importance of statistical leverage scores!

# Anecdote 2: Communities in large informatics graphs

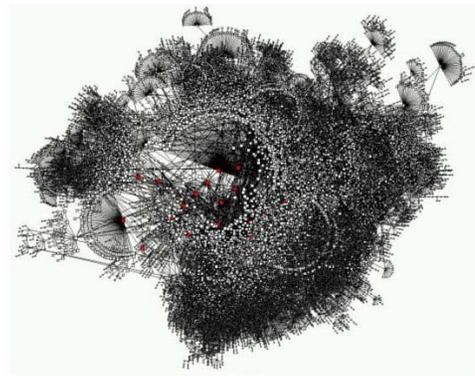
Mahoney "Algorithmic and Statistical Perspectives on Large-Scale Data Analysis" (2010)  
Leskovec, Lang, Dasgupta, & Mahoney "Community Structure in Large Networks ..." (2009)

**Data are expander-like  
at large size scales !!!**

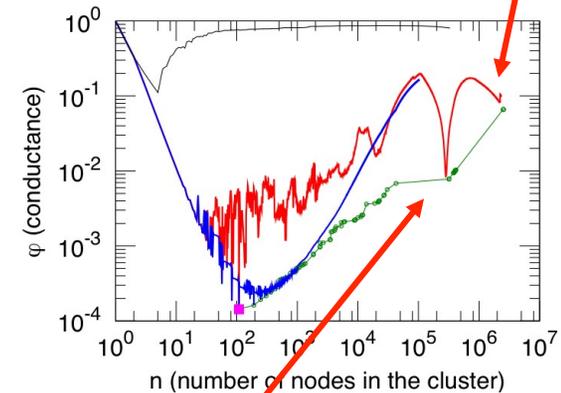
People imagine social  
networks to look like:



Real social networks  
actually look like:



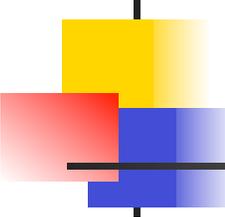
Size-resolved conductance  
(degree-weighted  
expansion) plot looks like:



How do we know this plot is "correct"?

- (since computing conductance is intractable)
- Algorithmic Result (ensemble of sets returned by different approximation algorithms are very different)
- Statistical Result (Spectral provides more meaningful communities than flow)
- Lower Bound Result; Structural Result; Modeling Result; Etc.

**There do not exist good large  
clusters in these graphs !!!**



# Lessons from the anecdotes

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Mahoney "Algorithmic and Statistical Perspectives on Large-Scale Data Analysis" (2010)

We are being forced to **engineer a union between two very different worldviews** on what are fruitful ways to view the data

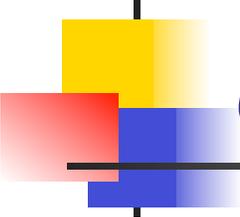
- in spite of our best efforts *not* to

Often fruitful to consider the **statistical properties implicit in worst-case algorithms**

- rather that *first* doing statistical modeling and *then* doing applying a computational procedure as a black box
- for both anecdotes, this was *essential* for leading to "useful theory"

How to extend these ideas to "bridge the gap" b/w the theory and practice of **MMDS** (Modern Massive Data Set) analysis.

- **QUESTION:** *Can we identify a/the concept at the heart of the algorithmic-statistical disconnect and then drill-down on it?*



# Outline and overview

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Preamble: algorithmic & statistical perspectives

General thoughts: data, algorithms, and explicit & implicit regularization

Approximate first nontrivial eigenvector of Laplacian

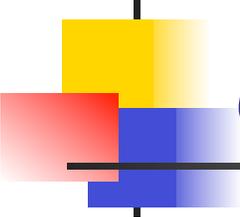
- Three random-walk-based procedures (heat kernel, PageRank, truncated lazy random walk) are *implicitly solving a regularized optimization exactly!*

Spectral versus flow-based algs for graph partitioning

- Theory says *each regularizes in different ways*; empirical results agree!

Weakly-local and strongly-local graph partitioning methods

- *Operationally like L1-regularization and already used in practice!*



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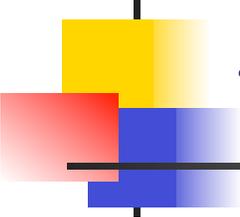
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# Thoughts on models of data (1 of 2)

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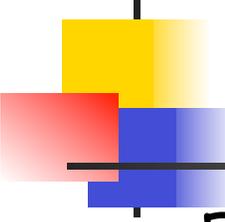
## Data are whatever data are

- records of banking/financial transactions, hyperspectral medical/astronomical images, electromagnetic signals in remote sensing applications, DNA microarray/SNP measurements, term-document data, search engine query/click logs, user interactions on social networks, corpora of images, sounds, videos, etc.

To do something useful, *you must model the data*

## Two criteria when choosing a data model

- (data acquisition/generation side): want a structure that is "close enough" to the data that you don't do too much "damage" to the data
- (downstream/analysis side): want a structure that is at a "sweet spot" between descriptive flexibility and algorithmic tractability



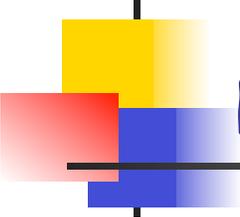
## Thoughts on models of data (2 of 2)

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Examples of data models:

- *Flat tables and the relational model*: one or more two-dimensional arrays of data elements, where different arrays can be related by predicate logic and set theory.
- *Graphs, including trees and expanders*:  $G=(V,E)$ , with a set of nodes  $V$  that represent "entities" and edges  $E$  that represent "interactions" between pairs of entities.
- *Matrices, including SPSD matrices*:  $m$  "objects," each of which is described by  $n$  "features," i.e., an  $n$ -dimensional Euclidean vector, gives an  $m \times n$  matrix  $A$ .

*Much modern data are relatively-unstructured; matrices and graphs are often useful, especially when traditional databases have problems.*



## Relationship b/w algorithms and data (1 of 3)

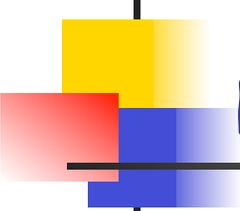
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### Before the digital computer:

- **Natural sciences** rich source of problems, **statistical methods** developed to solve those problems
- *Very important notion: **well-posed (well-conditioned) problem**: solution exists, is unique, and is continuous w.r.t. problem parameters*
- *Simply doesn't make sense to solve ill-posed problems*

### Advent of the digital computer:

- **Split in** (yet-to-be-formed field of) "**Computer Science**"
- **Based on application** (scientific/numerical computing vs. business/consumer applications) **as well as tools** (continuous math vs. discrete math)
- *Two very different perspectives on relationship b/w algorithms and data*



## Relationship b/w algorithms and data (2 of 3)

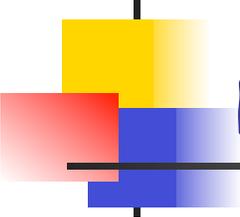
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### Two-step approach for "numerical" problems

- Is problem well-posed/well-conditioned?
- If no, replace it with a well-posed problem. (Regularization!)
- If yes, design a stable algorithm.

### View Algorithm A as a function $f$

- Given  $x$ , it tries to compute  $y$  but actually computes  $y^*$
- **Forward error**:  $\Delta y = y^* - y$
- **Backward error**: smallest  $\Delta x$  s.t.  $f(x + \Delta x) = y^*$
- Forward error  $\leq$  Backward error \* condition number
- *Backward-stable algorithm provides accurate solution to well-posed problem!*



## Relationship b/w algorithms and data (3 of 3)

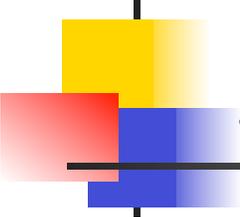
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### One-step approach for study of computation, *per se*

- Concept of computability captured by 3 seemingly-different discrete processes (recursion theory,  $\lambda$ -calculus, Turing machine)
- Computable functions have internal structure (P vs. NP, NP-hardness, etc.)
- Problems of practical interest are “intractable” (e.g., NP-hard vs. poly(n), or  $O(n^3)$  vs.  $O(n \log n)$ )

### Modern Theory of Approximation Algorithms

- provides **forward-error** bounds for **worst-case** input
- worst case in two senses: (1) for all possible input & (2) i.t.o. relatively-simple complexity measures, but independent of “structural parameters”
- get bounds by “**relaxations**” of IP to LP/SDP/etc., i.e., a “**nicer**” place



## Statistical regularization (1 of 3)

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### Regularization in statistics, ML, and data analysis

- arose in integral equation theory to “solve” ill-posed problems
- computes a **better or more “robust” solution**, so better inference
- involves making (explicitly or implicitly) assumptions about data
- provides a **trade-off between “solution quality” versus “solution niceness”**
- often, heuristic approximation procedures have regularization properties as a “side effect”
- lies at *the heart of the disconnect between the “algorithmic perspective” and the “statistical perspective”*

## Statistical regularization (2 of 3)

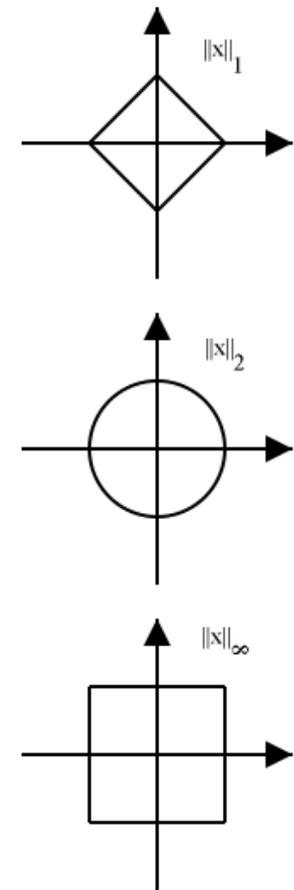
Usually *implemented* in 2 steps:

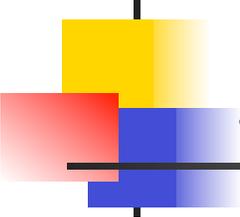
- add a norm constraint (or “geometric capacity control function”)  $g(x)$  to objective function  $f(x)$
- solve the modified optimization problem

$$x' = \operatorname{argmin}_x f(x) + \lambda g(x)$$

Often, this is a “harder” problem, e.g., L1-regularized L2-regression

$$x' = \operatorname{argmin}_x \|Ax - b\|_2 + \lambda \|x\|_1$$





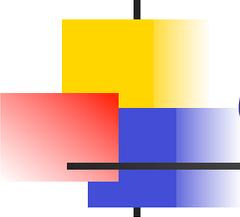
## Statistical regularization (3 of 3)

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**Regularization** is often observed as a side-effect or by-product of other **design decisions**

- “binning,” “pruning,” etc.
- “truncating” small entries to zero, “early stopping” of iterations
- approximation algorithms and **heuristic approximations engineers do to implement algorithms in large-scale systems**

**BIG question:** *Can we formalize the notion that/when approximate computation can **implicitly** lead to “better” or “more regular” solutions than exact computation?*



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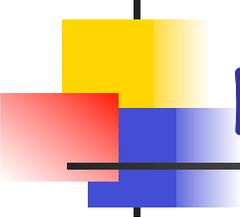
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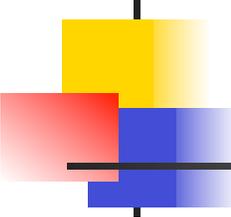
- Operationally like L1-regularization and already used in practice!



## Notation for weighted undirected graph

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- vertex set  $V = \{1, \dots, n\}$
- edge set  $E \subset V \times V$
- edge weight function  $w : E \rightarrow R_+$
- degree function  $d : V \rightarrow R_+$ ,  $d(u) = \sum_v w(u, v)$
- diagonal degree matrix  $D \in R^{V \times V}$ ,  $D(v, v) = d(v)$
- combinatorial Laplacian  $L_0 = D - W$
- normalized Laplacian  $L = D^{-1/2} L_0 D^{-1/2}$



# Approximating the top eigenvector

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**Basic idea:** Given an SPSD (e.g., Laplacian) matrix  $A$ ,

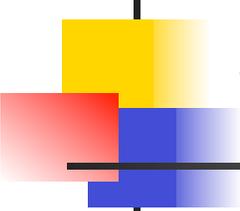
- **Power method** starts with  $v_0$ , and iteratively computes

$$v_{t+1} = Av_t / \|Av_t\|_2 .$$

- Then,  $v_t = \sum_i \gamma_i^t v_i \rightarrow v_1$  .
- If we truncate after (say) 3 or 10 iterations, still have some mixing from other eigen-directions

What **objective** does the exact eigenvector optimize?

- Rayleigh quotient  $R(A,x) = x^T A x / x^T x$ , for a *vector*  $x$ .
- But can also express this as an SDP, for a SPSD *matrix*  $X$ .
- (We will **put regularization on this SDP!**)



# Views of approximate spectral methods

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Three common procedures ( $L$ =Laplacian, and  $M$ =r.w. matrix):

- Heat Kernel:

$$H_t = \exp(-tL) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} L^k$$

- PageRank:

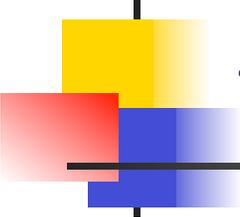
$$\pi(\gamma, s) = \gamma s + (1 - \gamma) M \pi(\gamma, s)$$

$$R_\gamma = \gamma (I - (1 - \gamma) M)^{-1}$$

- $q$ -step Lazy Random Walk:

$$W_\alpha^q = (\alpha I + (1 - \alpha) M)^q$$

Question: Do these "approximation procedures" exactly optimizing some regularized objective?



## Two versions of spectral partitioning

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**VP:**

$$\min. \quad x^T L_G x$$

$$\text{s.t.} \quad x^T L_{K_n} x = 1$$

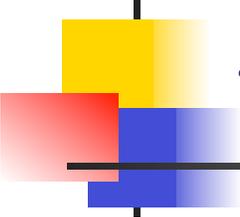
$$\langle x, 1 \rangle_D = 0$$



**R-VP:**

$$\min. \quad x^T L_G x + \lambda f(x)$$

$$\text{s.t.} \quad \textit{constraints}$$



## Two versions of spectral partitioning

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**VP:**

$$\begin{aligned} \min. \quad & x^T L_G x \\ \text{s.t.} \quad & x^T L_{K_n} x = 1 \\ & \langle x, 1 \rangle_D = 0 \end{aligned}$$



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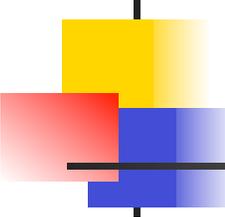
**SDP:**

$$\begin{aligned} \min. \quad & L_G \circ X \\ \text{s.t.} \quad & L_{K_n} \circ X = 1 \\ & X \succeq 0 \end{aligned}$$



**R-SDP:**

$$\begin{aligned} \min. \quad & L_G \circ X + \lambda F(X) \\ \text{s.t.} \quad & \text{constraints} \end{aligned}$$



# A simple theorem

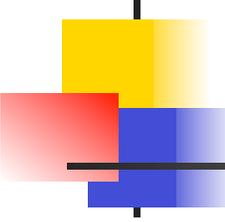
Mahoney and Orecchia (2010)

$$\begin{aligned} (\mathbf{F}, \eta)\text{-SDP} \quad & \min \quad L \bullet X + \frac{1}{\eta} \cdot F(X) \\ & \text{s.t.} \quad I \bullet X = 1 \\ & \quad \quad X \succeq 0 \end{aligned}$$

Modification of the usual SDP form of spectral to have regularization (but, on the matrix  $X$ , not the vector  $x$ ).

**Theorem:** Let  $G$  be a connected, weighted, undirected graph, with normalized Laplacian  $L$ . Then, the following conditions are sufficient for  $X^*$  to be an optimal solution to  $(\mathbf{F}, \eta)$ -SDP.

- $X^* = (\nabla F)^{-1} (\eta \cdot (\lambda^* I - L))$ , for some  $\lambda^* \in \mathbb{R}$ ,
- $I \bullet X^* = 1$ ,
- $X^* \succeq 0$ .



## Three simple corollaries

---

$F_H(X) = \text{Tr}(X \log X) - \text{Tr}(X)$  (i.e., generalized entropy)

gives scaled Heat Kernel matrix, with  $t = \eta$

$F_D(X) = -\log \det(X)$  (i.e., Log-determinant)

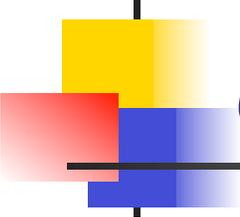
gives scaled PageRank matrix, with  $t \sim \eta$

$F_p(X) = (1/p) \|X\|_p^p$  (i.e., matrix p-norm, for  $p > 1$ )

gives Truncated Lazy Random Walk, with  $\lambda \sim \eta$

*(  $F(\bullet)$  specifies the algorithm; "number of steps" specifies the  $\eta$  )*

**Answer: These "approximation procedures" compute regularized versions of the Fiedler vector *exactly!***



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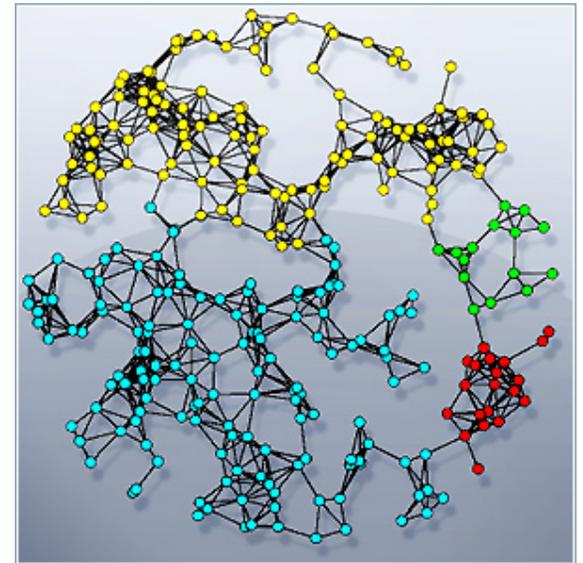
# Graph partitioning

A family of combinatorial optimization problems - want to partition a graph's nodes into two sets s.t.:

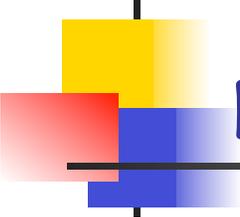
- Not much edge weight across the cut (cut quality)
- Both sides contain a lot of nodes

Several standard formulations:

- Graph bisection (minimum cut with 50-50 balance)
- $\beta$ -balanced bisection (minimum cut with 70-30 balance)
- $\text{cutsize}/\min\{|A|,|B|\}$ , or  $\text{cutsize}/(|A||B|)$  (expansion)
- $\text{cutsize}/\min\{\text{Vol}(A),\text{Vol}(B)\}$ , or  $\text{cutsize}/(\text{Vol}(A)\text{Vol}(B))$  (conductance or N-Cuts)



All of these formalizations of the bi-criterion are NP-hard!



# Networks and networked data

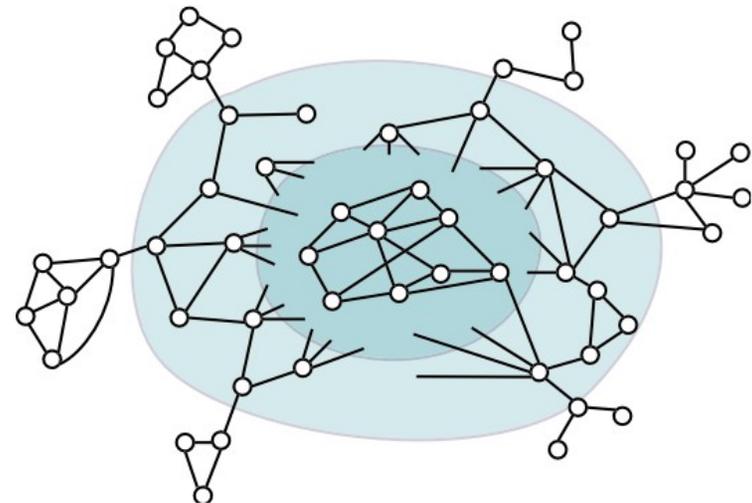
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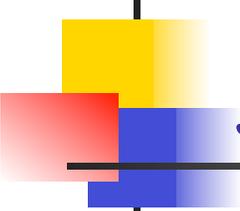
## Lots of “networked” data!!

- technological networks
  - AS, power-grid, road networks
- biological networks
  - food-web, protein networks
- social networks
  - collaboration networks, friendships
- information networks
  - co-citation, blog cross-postings, advertiser-bidder phrase graphs...
- language networks
  - semantic networks...
- ...

## Interaction graph model of networks:

- **Nodes** represent “entities”
- **Edges** represent “interaction” between pairs of entities





# Social and Information Networks

• Social nets	Nodes	Edges	Description
LIVEJOURNAL	4,843,953	42,845,684	Blog friendships [4]
EPINIONS	75,877	405,739	Who-trusts-whom [35]
FLICKR	404,733	2,110,078	Photo sharing [21]
DELICIOUS	147,567	301,921	Collaborative tagging
CA-DBLP	317,080	1,049,866	Co-authorship (CA) [4]
CA-COND-MAT	21,363	91,286	CA cond-mat [25]
• Information networks			
CIT-HEP-TH	27,400	352,021	hep-th citations [13]
BLOG-POSTS	437,305	565,072	Blog post links [28]
• Web graphs			
WEB-GOOGLE	855,802	4,291,352	Web graph Google
WEB-WT10G	1,458,316	6,225,033	TREC WT10G web
• Bipartite affiliation (authors-to-papers) networks			
ATP-DBLP	615,678	944,456	DBLP [25]
ATP-ASTRO-PH	54,498	131,123	Arxiv astro-ph [25]
• Internet networks			
AS	6,474	12,572	Autonomous systems
GNUTELLA	62,561	147,878	P2P network [36]

**Table 1: Some of the network datasets we studied.**

# Motivation: Sponsored (“paid”) Search

Text based ads driven by user specified query

The process:

- Advertisers bids on query phrases.
- Users enter query phrase.
- Auction occurs.
- Ads selected, ranked, displayed.
- When user clicks, advertiser pays!

The screenshot shows a Yahoo! search results page for the query "barcelona chair". The search bar at the top contains the text "barcelona chair" and a "Search" button. The page displays 1-10 of 4,220,000 results for "barcelona chair" in 0.09 seconds. Below the search bar, there are several sponsored results and organic search results.

**Sponsored Results:**

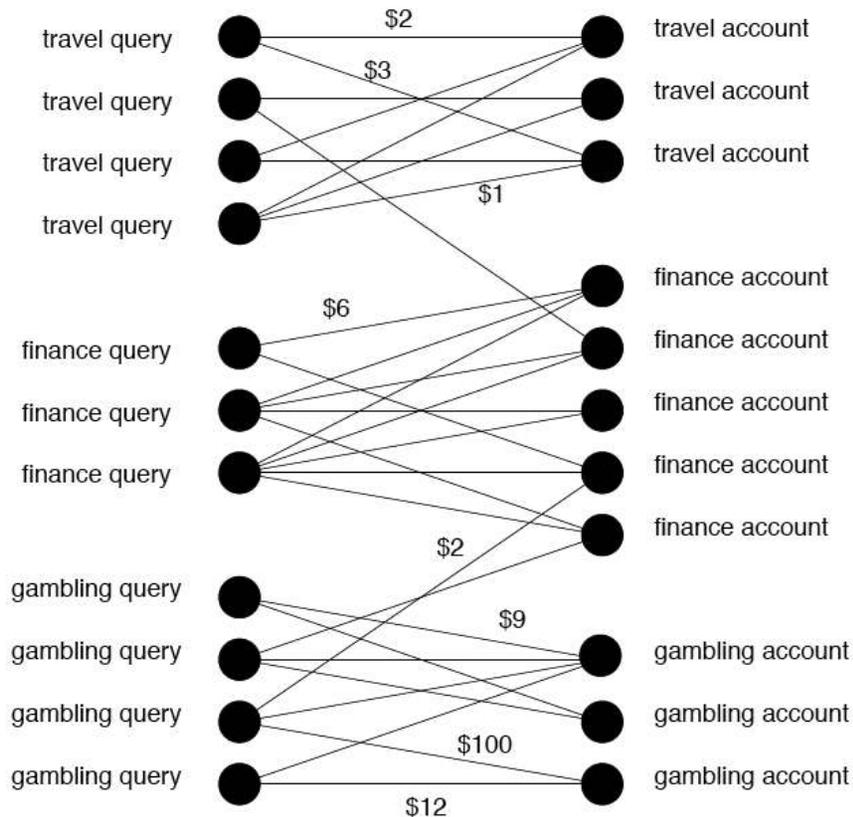
- Barcelona Chair Direct from Importer**  
Barcelona Sofa, Barcelona Chair and more Barcelona furniture designs.  
[www.WickedElements.com](http://www.WickedElements.com)
- Barcelona Chairs**  
Chairs & Seats from 152+ Shops. Barcelona Chairs on Sale.  
[www.Calibex.com](http://www.Calibex.com)
- Barcelona Chair - \$659.99 Free Shipping**  
Loveseat, daybed, ottoman. Free shipping. Up to 60% off.  
[www.modabode.com](http://www.modabode.com)
- Buy Barcelona Chairs**  
We Have 13,900+ Sofas. Barcelona Chairs on Sale.  
[www.NexTag.com/sofas](http://www.NexTag.com/sofas)
- Barcelona Chair**  
The Right Style For Your Space. Barcelona chair From \$20.  
[Shopzilla.com/chairs](http://Shopzilla.com/chairs)

**Organic Search Results:**

- Also try:** [barcelona style chair](#), [knoll barcelona chair](#), [More...](#)
- SPONSOR RESULTS**
  - Barcelona Chair: Sale Weekend**  
[www.PGMod.com/Barcelona-Chair](http://www.PGMod.com/Barcelona-Chair) - Customer Appreciation Sale! Save 5% on Barcelona Chair + Free S&H.
  - Barcelona Chair - Free Shipping**  
[www.moderncollections.com](http://www.moderncollections.com) - Avoid cheap imitations. Our Barcelona Chair offers genuine quality...
  - Barcelona Chairs**  
[BizRate.com](http://BizRate.com) - We Offer 2,500+ Chair Choices. Deals On barcelona chairs.
  - Classic Barcelona Chair On Sale \$899**  
[funkysofa.com](http://funkysofa.com) - All colors available. The Barcelona Chair is a classic piece that...
- 1. Barcelona Chair - Volo Leather**  
Ludwig Mies van der Rohe's Barcelona Chair and Stool (1929), originally created to furnish his German Pavilion at the International Exhibition in Barcelona, have come...  
[www.dwr.com/productdetail.cfm?id=7200](http://www.dwr.com/productdetail.cfm?id=7200) - 17k
- 2. Barcelona chair - Wikipedia, the free encyclopedia**  
The Barcelona chair and ottoman was designed by Mies van der Rohe for ... Barcelona Chair, inspired by its predecessors, the campaign and folding chairs ...

**Feedback:** Yahoo!s: [Report](#) bad results or ads. Bucket test: F655

# Bidding and Spending Graphs



Uses of Bidding and Spending graphs:

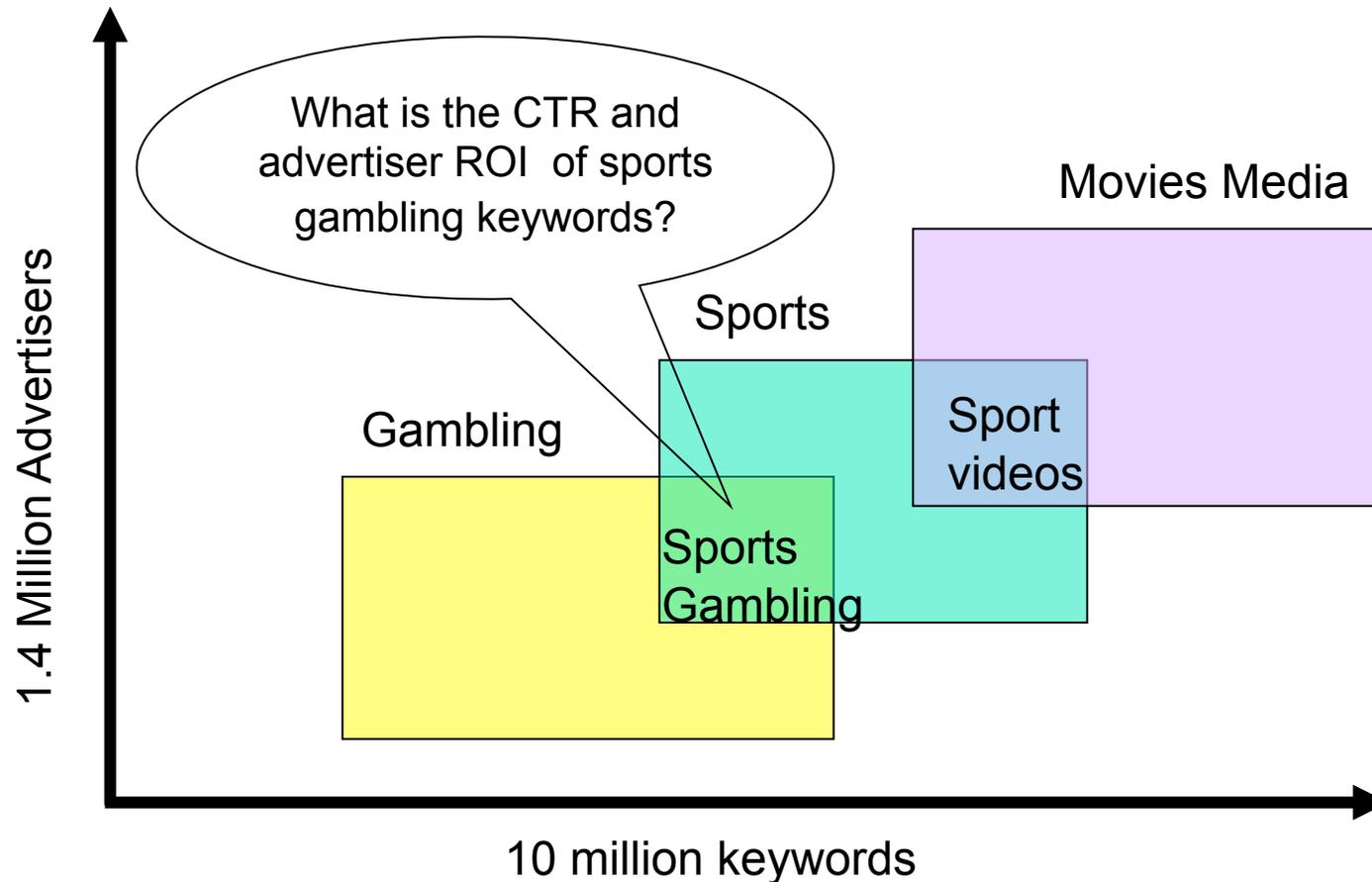
- “deep” micro-market identification.
- improved query expansion.

More generally, user segmentation for behavioral targeting.

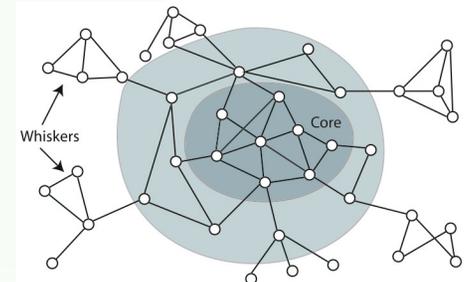
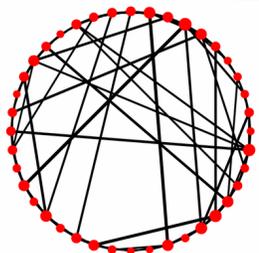
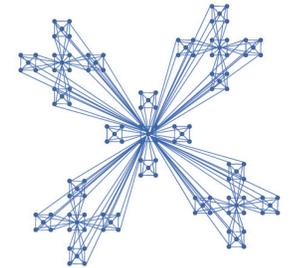
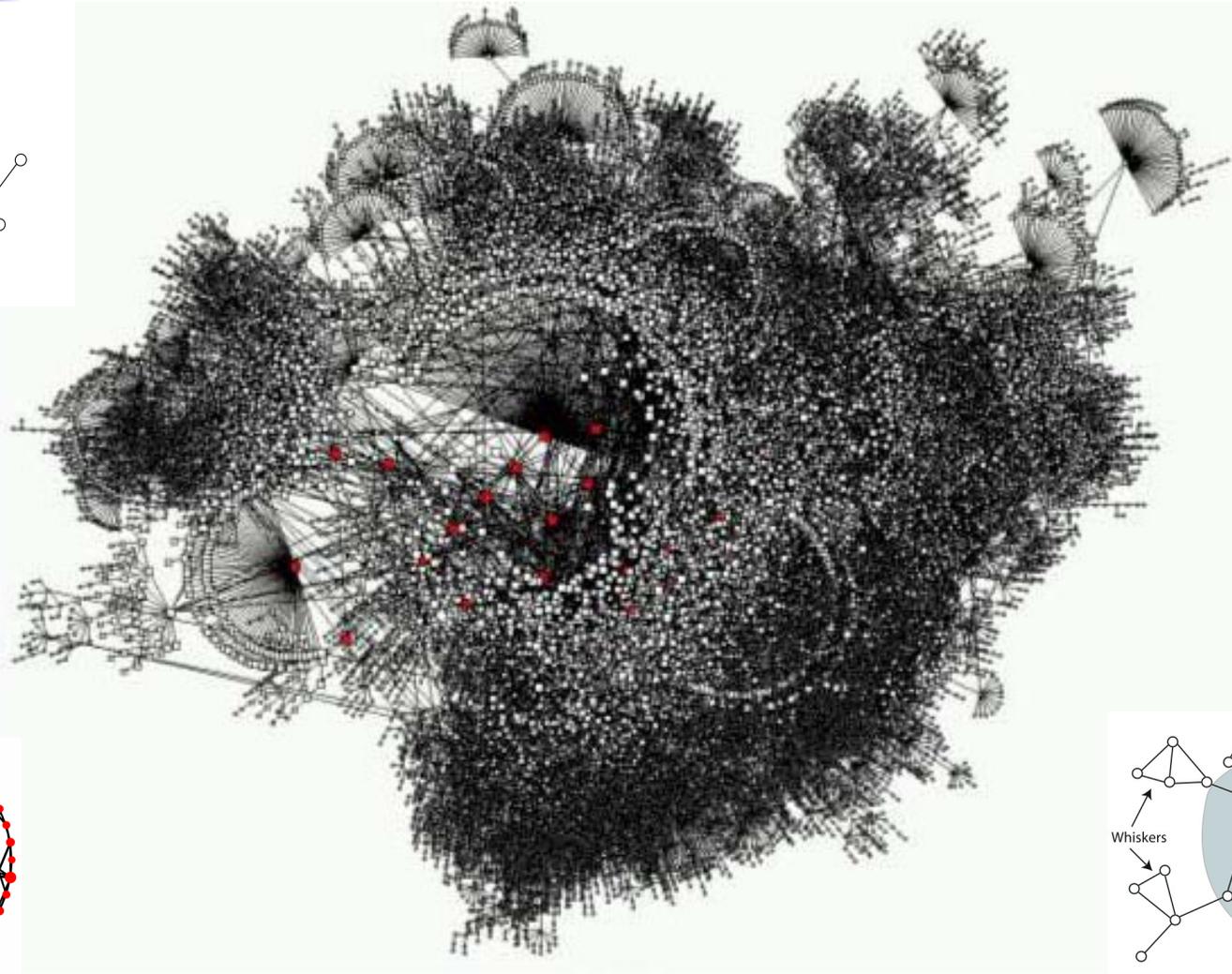
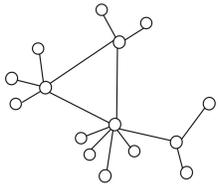
A “social network” with “term-document” aspects.

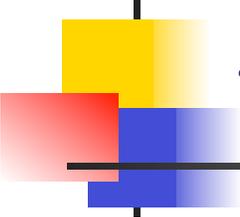
# Micro-markets in sponsored search

Goal: Find *isolated* markets/clusters with *sufficient money/clicks* with *sufficient coherence*.  
Ques: Is this even possible?



# What do these networks "look" like?





## The "lay of the land"

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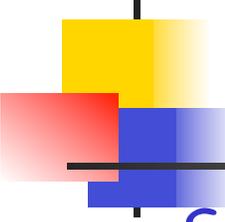
**Spectral methods\*** - compute eigenvectors of associated matrices

**Local improvement** - easily get trapped in local minima, but can be used to clean up other cuts

**Multi-resolution** - view (typically space-like graphs) at multiple size scales

**Flow-based methods\*** - single-commodity or multi-commodity version of max-flow-min-cut ideas

\*Comes with *strong* underlying theory to guide heuristics.



## Comparison of "spectral" versus "flow"

---

### Spectral:

- Compute an eigenvector
- "Quadratic" worst-case bounds
- Worst-case achieved -- on "long stringy" graphs
- Worst-case is "local" property
- Embeds you on a line (or  $K_n$ )

### Flow:

- Compute a LP
- $O(\log n)$  worst-case bounds
- Worst-case achieved -- on expanders
- Worst case is "global" property
- Embeds you in  $L_1$

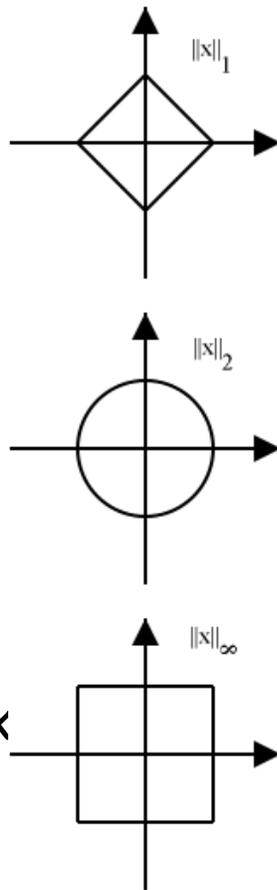
Two methods -- complementary strengths and weaknesses

- What we compute is determined at least as much by as the approximation algorithm as by objective function.

# Explicit versus implicit geometry

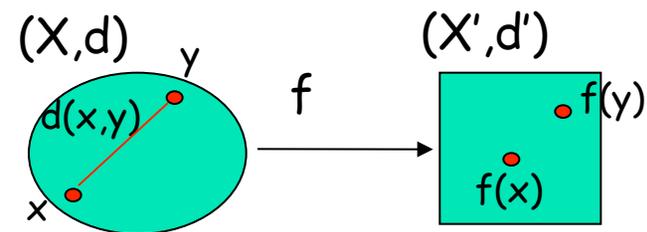
## Explicitly-imposed geometry

- Traditional regularization uses *explicit* norm constraint to make sure solution vector is "small" and not-too-complex

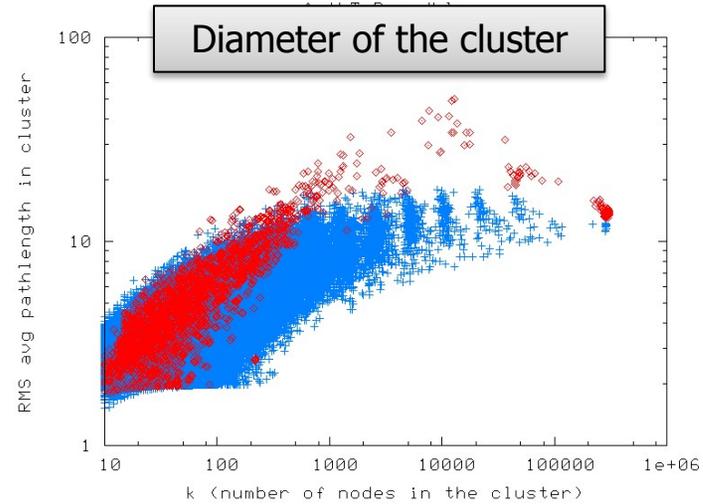
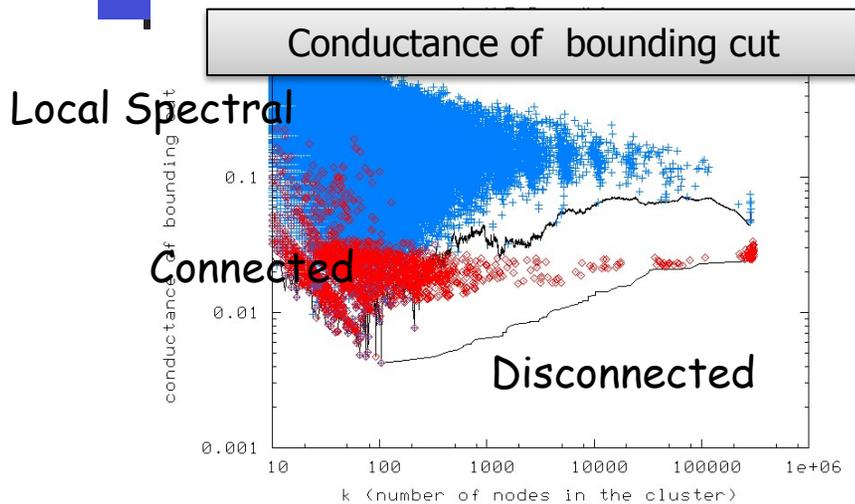


## Implicitly-imposed geometry

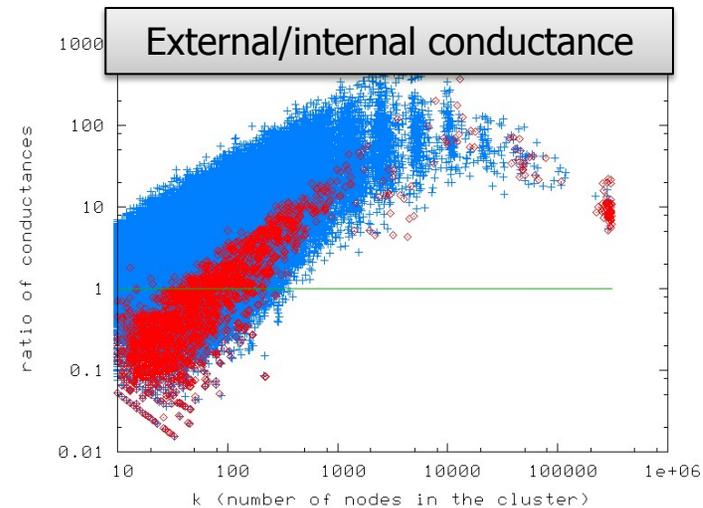
- Approximation algorithms *implicitly* embed the data in a "nice" metric/geometric place and then round the solution.



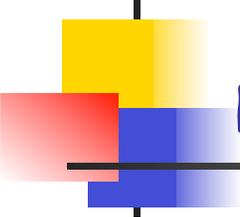
# Regularized and non-regularized communities (1 of 2)



- **Metis+MQI - a Flow-based method (red)** gives sets with better conductance.
- **Local Spectral (blue)** gives tighter and more well-rounded sets.



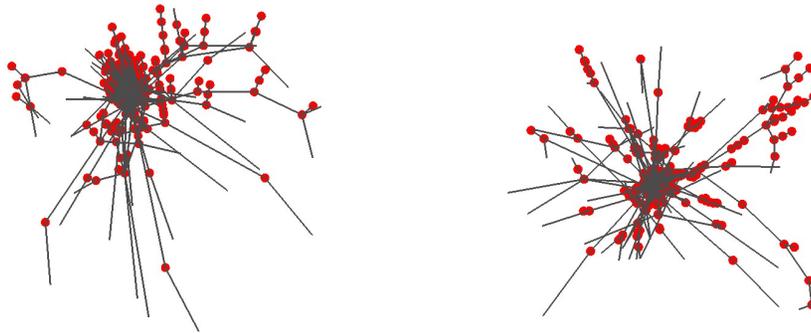
Lower is good



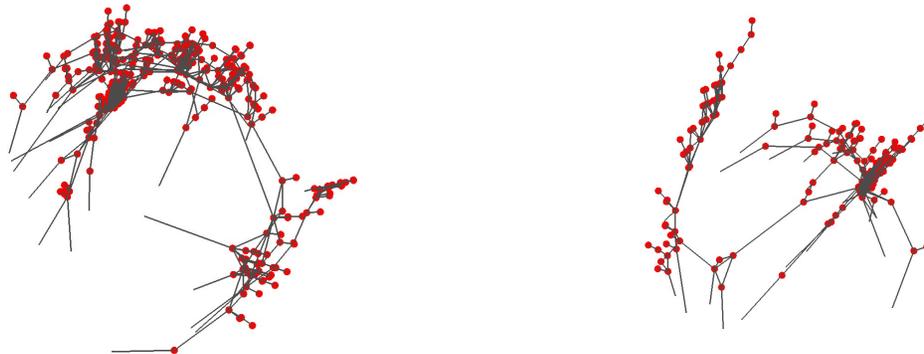
## Regularized and non-regularized communities (2 of 2)

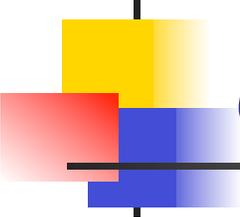
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Two ca. 500 node communities from Local Spectral Algorithm:



Two ca. 500 node communities from Metis+MQI:





# Outline and overview

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Preamble: algorithmic & statistical perspectives

General thoughts: data, algorithms, and explicit & implicit regularization

Approximate first nontrivial eigenvector of Laplacian

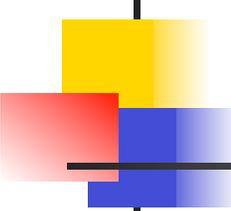
- Three random-walk-based procedures (heat kernel, PageRank, truncated lazy random walk) are *implicitly* solving a regularized optimization *exactly!*

Spectral versus flow-based algs for graph partitioning

- Theory says each regularizes in different ways; empirical results agree!

Weakly-local and strongly-local graph partitioning methods

- **Operationally like L1-regularization, and already used in practice!**

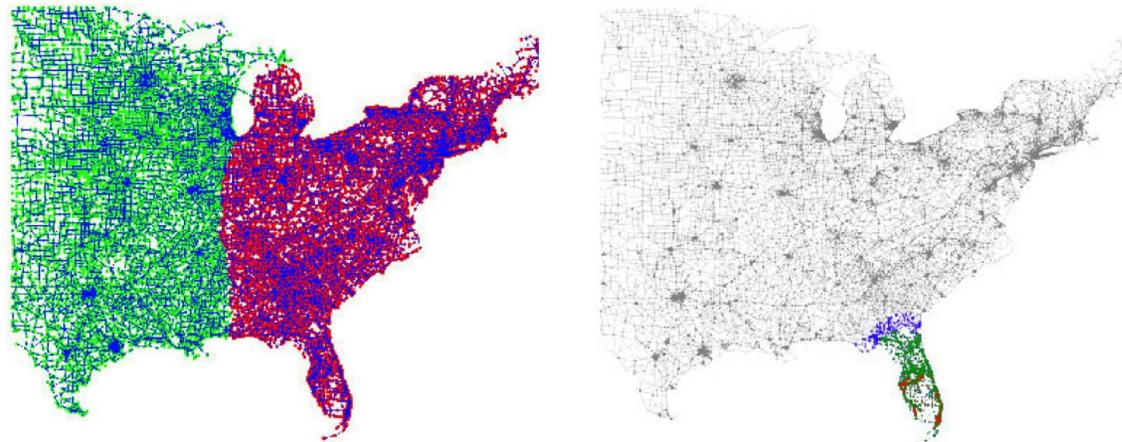


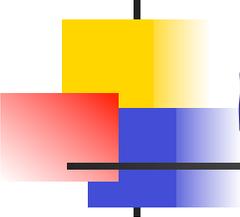
# Computing locally-biased partitions

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Often want clusters “near” a pre-specified set of nodes:

- Large social graphs have good small clusters, don’t have good large clusters
- Might have domain knowledge, so find “semi-supervised” clusters
- As algorithmic primitives, e.g., to solve linear equations fast.





# Recall global spectral graph partitioning

The basic optimization problem:

$$\begin{array}{ll} \text{minimize} & x^T L_G x \\ \text{s.t.} & \langle x, x \rangle_D = 1 \\ & \langle x, \mathbf{1} \rangle_D = 0 \end{array}$$

- Relaxation of:

$$\phi(G) = \min_{S \subset V} \frac{E(S, \bar{S})}{\text{Vol}(S)\text{Vol}(\bar{S})}$$

- Solvable via the eigenvalue problem:

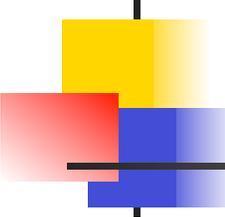
$$\mathcal{L}_G y = \lambda_2(G) y$$

- Sweep cut of second eigenvector yields:

$$\lambda_2(G)/2 \leq \phi(G) \leq \sqrt{8\lambda_2(G)}$$

Idea to compute locally-biased partitions:

- Modify this objective with a locality constraint
- Show that some/all of these nice properties still hold locally



# Local spectral partitioning *ansatz*

Mahoney, Orecchia, and Vishnoi (2010)

## Primal program:

$$\begin{aligned} \text{minimize} \quad & x^T L_G x \\ \text{s.t.} \quad & \langle x, x \rangle_D = 1 \\ & \langle x, s \rangle_D^2 \geq \kappa \end{aligned}$$

## Interpretation:

- Find a cut well-correlated with the seed vector  $s$ .
- If  $s$  is a single node, this relaxes:

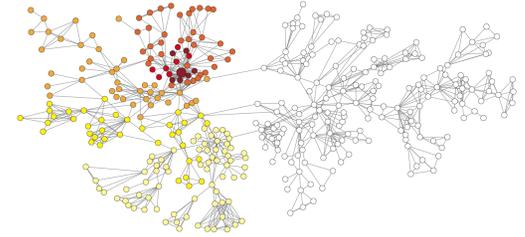
$$\min_{S \subset V, s \in S, |S| \leq 1/k} \frac{E(S, \bar{S})}{\text{Vol}(S)\text{Vol}(\bar{S})}$$

## Dual program:

$$\begin{aligned} \text{max} \quad & \alpha - \beta(1 - \kappa) \\ \text{s.t.} \quad & L_G \succeq \alpha L_{K_n} - \beta \left( \frac{L_{K_T}}{\text{vol}(\bar{T})} + \frac{L_{K_{\bar{T}}}}{\text{vol}(T)} \right) \\ & \beta \geq 0 \end{aligned}$$

## Interpretation:

- Embedding a combination of scaled complete graph  $K_n$  and complete graphs  $T$  and  $\bar{T}$  ( $K_T$  and  $K_{\bar{T}}$ ) - where the latter encourage cuts near  $(T, \bar{T})$ .



# Main theoretical results

Mahoney, Orecchia, and Vishnoi (2010)

**Theorem:** If  $x^*$  is an optimal solution to LocalSpectral,

(\*) it is a **Generalized Personalized PageRank vector**, and can be computed as solution to a **set of linear equations**;

Fast running time guarantee.

(\*) one can find a cut of **conductance**  $\leq 8\lambda(G, s, \kappa)$  in time  $O(n \lg n)$  with sweep cut of  $x^*$ ;

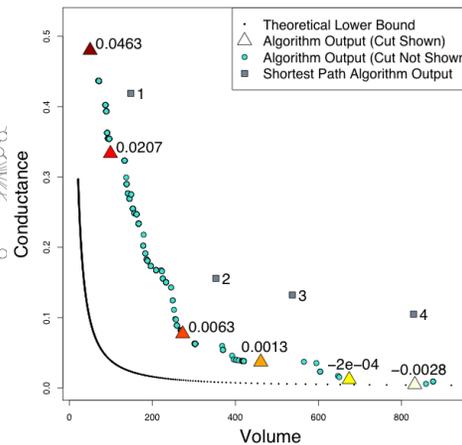
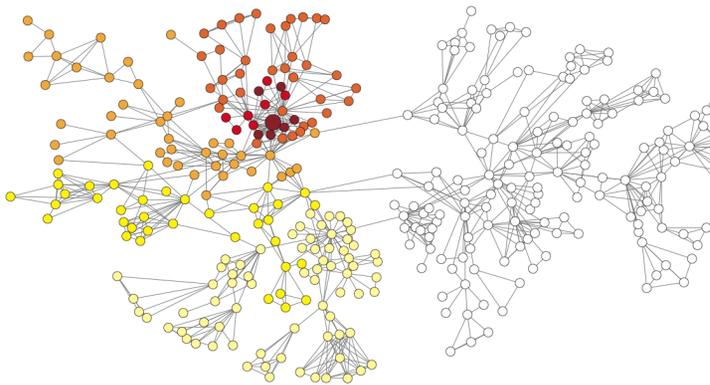
Upper bound, as usual from sweep cut & Cheeger.

(\*) For all sets of nodes  $T$  s.t.  $\kappa' := \langle s, s_T \rangle_D^2$ , we have:  $\phi(T) \geq \lambda(G, s, \kappa)$  if  $\kappa \leq \kappa'$ , and  $\phi(T) \geq (\kappa'/\kappa)\lambda(G, s, \kappa)$  if  $\kappa' \leq \kappa$ .

Lower bound: Spectral version of flow-improvement algs.

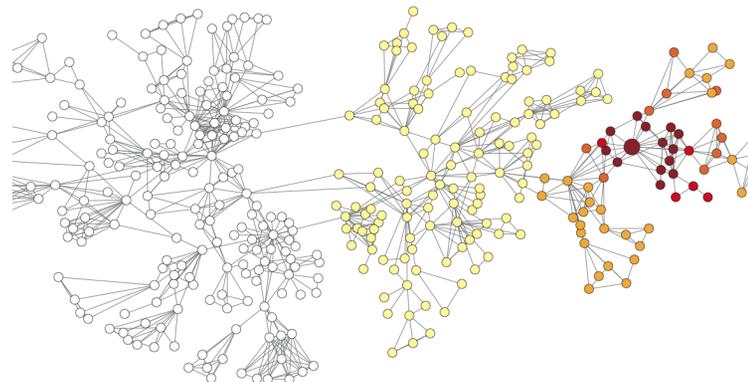
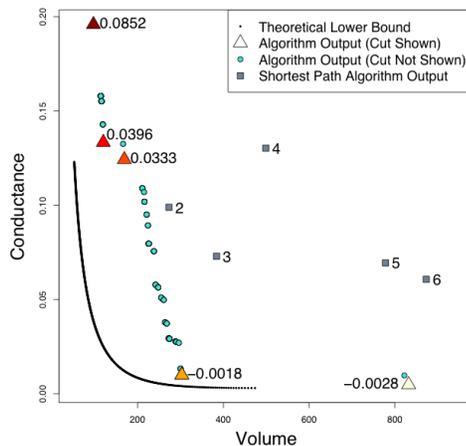
# Illustration on small graphs

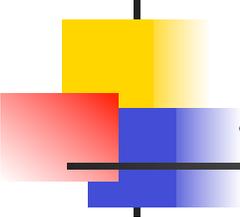
Mahoney, Orecchia, and Vishnoi (2010)



- Similar results if we do local random walks, truncated PageRank, and heat kernel diffusions.

- Often, it finds "worse" quality but "nicer" partitions than flow-improve methods. (Tradeoff we'll see later.)





# A somewhat different approach

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## Strongly-local spectral methods

ST04: truncated "local" random walks to compute locally-biased cut

ACL06: approximate locally-biased PageRank vector computations

Chung08: approximate heat-kernel computation to get a vector

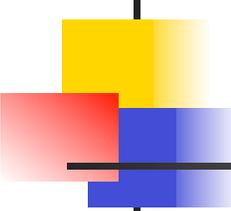
These are the diffusion-based procedures

that we saw before

*except truncate/round/clip/push small things to zero*

starting with localized initial condition

Also get provably-good local version of global spectral



## What's the connection?

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"Optimization" approach:

- Well-defined objective  $f$
- Weakly local (touch all nodes), so good for medium-scale problems
- Easy to use

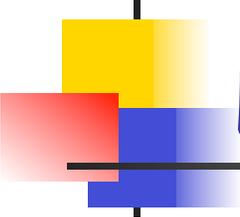
"Operational" approach\*:

- *Very* fast algorithm
- Strongly local (clip/truncate small entries to zero), good for large-scale
- Very difficult to use

\* *Informally, optimize  $f+\lambda g$  (... almost formally!): steps are structurally-similar to the steps of how, e.g., L1-regularized L2 regression algorithms, implement regularization*

More importantly,

- This "operational" approach is *already* being adopted in PODS/VLDB/SIGMOD/KDD/WWW environments!
- *Let's make the regularization explicit—and know what we compute!*



## Looking forward ...

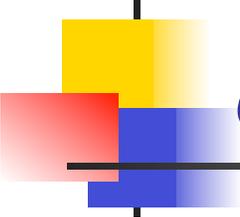
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A common *modus operandi* in many (really\*) large-scale applications is:

- Run a procedure that bears some resemblance to the procedure you would run if you were to solve a given problem exactly
- Use the output in a way similar to how you would use the exact solution, or prove some result that is similar to what you could prove about the exact solution.

**BIG Question:** Can we make this more principled? E.g., can we “engineer” the approximations to solve (exactly but implicitly) some regularized version of the original problem---to do large scale analytics in a statistically more principled way?

\*e.g., industrial production, publication venues like WWW, SIGMOD, VLDB, etc.



# Conclusions

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Regularization is:

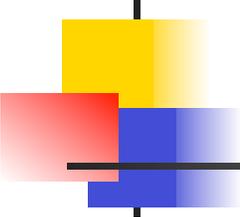
- absent from CS, which historically has studied computation per se
- central to nearly area that applies algorithms to noisy data
- gets at the heart of the algorithmic-statistical “disconnect”

Approximate computation, *in and of itself*, can implicitly regularize:

- Theory & the empirical signatures in matrix *and* graph problems
- Solutions of approximation algorithms don't need to be something we “settle for,” they can be “better” than the “exact” solution

In very large-scale analytics applications:

- Can we “engineer” database operations so “worst-case” approximation algorithms exactly solve regularized versions of original problem?
- I.e., can we get best of both worlds for very large-scale analytics?



# MADS Workshop on “Algorithms for Modern Massive Data Sets” (<http://mmds.stanford.edu>)

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at Stanford University, July 10-13, 2012

## Objectives:

- Address algorithmic, statistical, and mathematical challenges in modern statistical data analysis.
- Explore novel techniques for modeling and analyzing massive, high-dimensional, and nonlinearly-structured data.
- Bring together computer scientists, statisticians, mathematicians, and data analysis practitioners to promote cross-fertilization of ideas.

Organizers: M. W. Mahoney, A. Shkolnik, G. Carlsson, and P. Drineas,

*Registration is available now!*