Improved Algorithms for the Bichromatic Two-Center Problem for Pairs of Points

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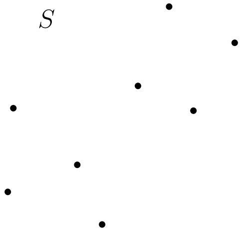
Background

- **2-center problem in the plane**
  Given a set S of n points in the plane, find two disks D_1^* and D_2^* such that $S \subseteq D_1^* \cup D_2^*$ and $\max\{\mathrm{rad}(D_1^*), \mathrm{rad}(D_2^*)\}$ is minimized.



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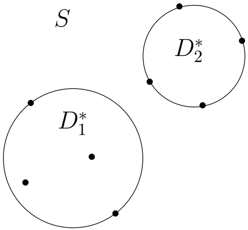

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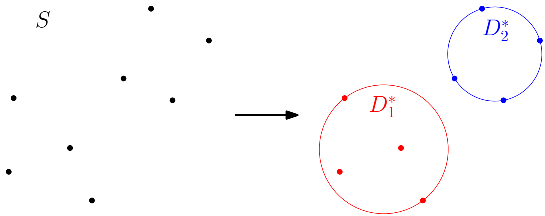
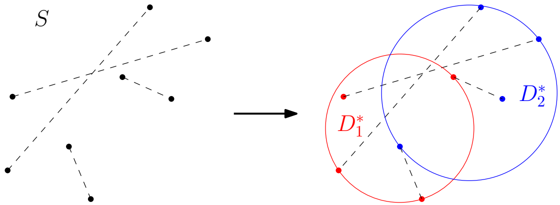
- **An equivalent definition**
  Color each point in S as red or blue such that $\max\{\mathrm{rad}(D_1^*), \mathrm{rad}(D_2^*)\}$ is minimized where D_1^* (resp., D_2^*) is the smallest enclosing disk of all red (resp., blue) points.



Problem definition

- **Bichromatic 2-center problem in the plane**
  Given a set S of n pairs of points in the plane, for every pair, color one point as red and the other as blue such that $\max\{\text{rad}(D_1^*), \text{rad}(D_2^*)\}$ is minimized where D_1^* (resp., D_2^*) is the smallest enclosing disk of all red (resp., blue) points.



Previous work and our result

- **Previous results for planar 2-center**
  - $O(n^2 \log^3 n)$ time [Agarwal and Sharir, 1994]
  - $O(n^2)$ time [Jaromczyk and Kowaluk, 1994]
  - $O(n \log^9 n)$ time [Sharir, 1997]
  - $O(n \log^2 n)$ expected time [Eppstein, 1997]
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- **Previous results for planar bichromatic 2-center**
  - $O(n^3 \log^2 n)$ time [Arkin et al., 2015]
  - $(1+\epsilon)$-approximation algorithms [Arkin et al., 2015]
    - $O((n/\varepsilon^2) \log n \log(1/\varepsilon))$ time [Arkin et al., 2015]
    - $O(n + (1/\varepsilon)^6 \log^2(1/\varepsilon))$ time [Arkin et al., 2015]

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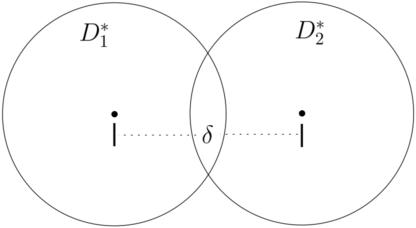
- **Our results for planar bichromatic 2-center**
  - $O(n^2 \log^2 n)$ time exact algorithm
  - $O(n + (1/\varepsilon)^3 \log^2(1/\varepsilon))$ time $(1+\epsilon)$-approximation

Exact algorithm

- Let D_1^* and D_2^* be the two disks of an optimal solution.
- Without loss of generality, we may assume that
  - D_1^* and D_2^* are congruent (let r^* denote their radius).
  - The distance δ between the centers of D_1^* and D_2^* is minimized.



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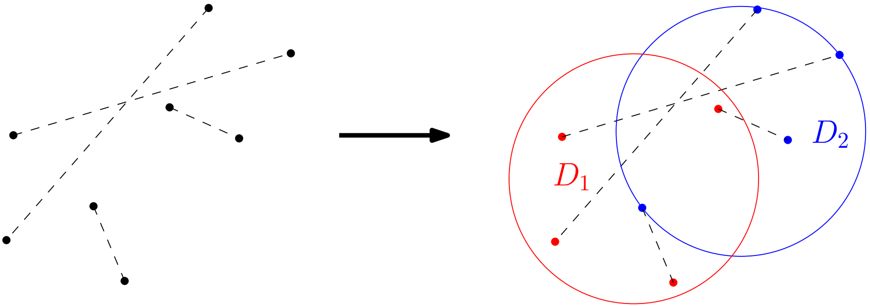
- **High-level idea**
  Distinguish two cases:
  - The distant case: $\delta \geq r^*$
  - The nearby case: $\delta < r^*$

  (Similar to the idea of [Sharir, 1997; Eppstein, 1997; Chan, 1999] for the planar 2-center problem)

A definition

Definition

We say a pair (D_1, D_2) of disks bichromatically covers S if it is possible to color a point as red and the other as blue for every pair of S such that D_1 (resp., D_2) covers all red (resp., blue) points.



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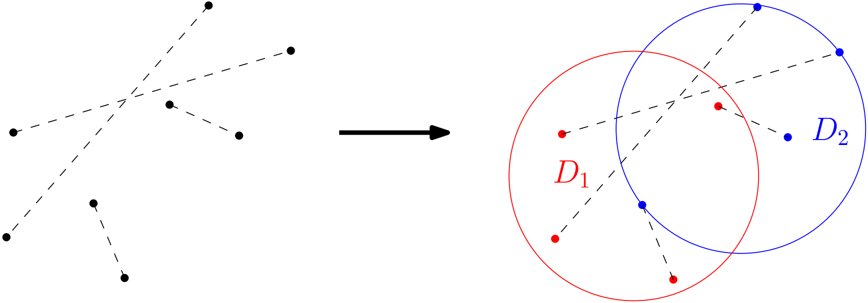


D_1 and D_2 are always congruent in our discussion

The distant case: $\delta \geq r^*$

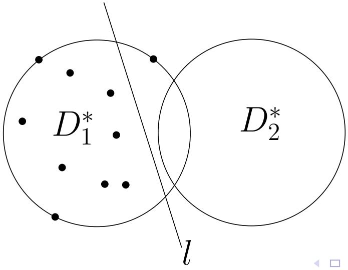
- **Basic strategy:** parametric search + decision
- **The decision problem**
  Given a value r, decide whether $r \geq r^*$, i.e., whether there exists a congruent pair of disks with radius r that bichromatically covers S.

The distant case

Observation (Eppstein, 1997)

One can determine in $O(n)$ time a set of $O(1)$ lines in which one line ℓ satisfies the following property.

- *The subset P_1 of all input points of S on the left side of ℓ are contained in one disk D_1^* of the optimal solution,*
- *At least one point of P_1 is on the boundary of D_1^**
- *D_1^* is the circurmcircle of two or three points of S.*



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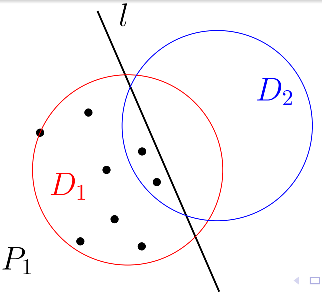
Lemma

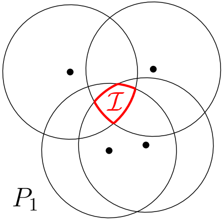
$r \geq r^*$ iff there exists a pair (D_1, D_2) of congruent disks of radius r bichromatically covering S with the following property.

- *All points in P_1 are contained in D_1*
- *At least one point of P_1 is on the boundary of D_1.*



The distant case

- $\mathcal{B}_r(a)$: the disk centered at a point a of radius r.
- $\mathcal{I} = \bigcap_{a \in P_1} \mathcal{B}_r(a)$



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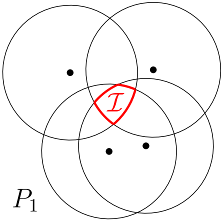

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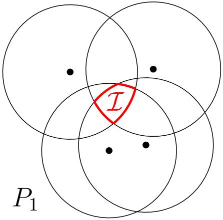

Lemma

D_1 satisfies the desired condition iff its center is on the boundary $\partial\mathcal{I}$ of $\mathcal{I}$.

- We say a point c is feasible if there exists (D_1, D_2) bichromatically covering S such that $D_1 = \mathcal{B}_r(c)$.
- It suffices to test the existence of a feasible point on $\partial\mathcal{I}$.

The distant case

Find a feasible point on $\partial\mathcal{I}$:

- For each point $c \in S \setminus P_1$, compute the (at most two) intersections $\partial\mathcal{I} \cap \partial\mathcal{B}_r(c)$.
- Q: the set of all such intersection points
- $|Q| = O(n)$

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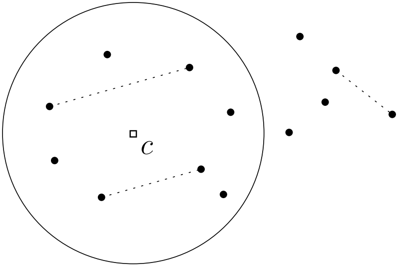
- $|Q| = O(n)$
- A feasible point exists on $\partial\mathcal{I}$ iff a feasible point exists in Q.
- For each point $c \in Q$, test whether it is a feasible point, i.e., whether there exists (D_1, D_2) bichromatically covering S such that $D_1 = \mathcal{B}_r(c)$.

The distant case

For each point $c \in Q$, test whether it is a feasible point:

- Check whether $\mathcal{B}_r(c)$ covers at least one point from each pair of S.



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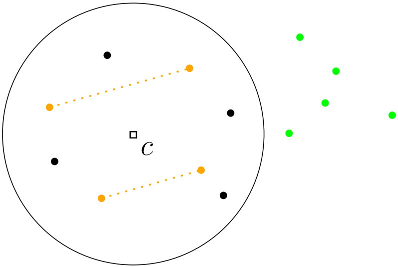
For each point $c \in Q$, test whether it is a feasible point:

- Check whether $\mathcal{B}_r(c)$ covers at least one point from each pair of S.
- Check whether there exists a disk of radius r covering all points of $P(c)$ and at least one point from each pair of $S(c)$
  - $P(c)$: points of S outside $\mathcal{B}_r(c)$
  - $S(c)$: pairs of S whose both points are in $\mathcal{B}_r(c)$



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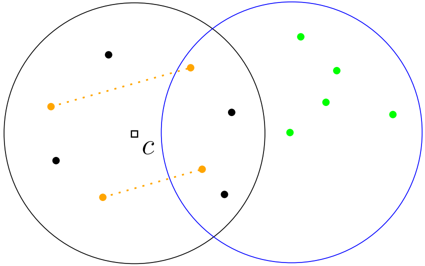

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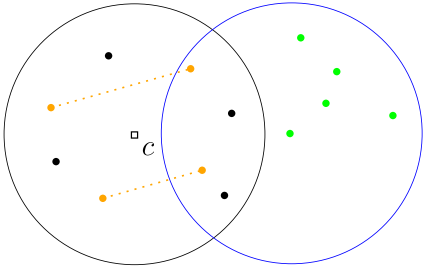


- $O(n \log n)$ time

The distant case

- Decision algorithm: $O(n^2 \log n)$ time

The distant case

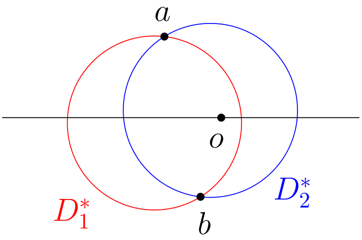
- Decision algorithm: $O(n^2 \log n)$ time
- Using Cole's parametric search, the optimization problem of the distant case can be solved in $O(n^2 \log^2 n)$ time.
  - Very similar to [Eppstein, 1997] for the planar 2-center problem.

The nearby case: $\delta < r^*$

- Consider the intersection $D_1^* \cap D_2^*$, which has two vertices a and b.

Observation (Eppstein, 1997)

In $O(n)$ time, one can find $O(1)$ points in which one point o is in $D_1^* \cap D_2^*$ and either the vertical or the horizontal line through o separates a and b.



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- Sort the points of S above (resp., below) ℓ counterclockwise around o.

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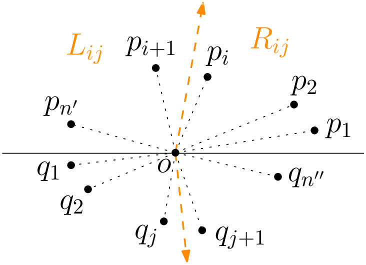

- Let $L_{i,j} = \{p_{i+1}, \ldots, p_{n'}, q_1, \ldots, q_j\}$ for $i \in [n']$ and $j \in [n'']$.
- Let $R_{i,j} = \{q_{j+1}, \ldots, q_{n''}, p_1, \ldots, p_i\}$ for $i \in [n']$ and $j \in [n'']$.

The nearby case

Lemma

For some $i \in [n']$ and $j \in [n'']$, $L_{i,j}$ is contained in one of D_1^* and D_2^* while $R_{i,j}$ is contained in the other.

The nearby case

Lemma

For some $i \in [n']$ *and* $j \in [n'']$, $L_{i,j}$ *is contained in one of* D_1^* *and* D_2^* *while* $R_{i,j}$ *is contained in the other.*

- Why is this true?

The nearby case

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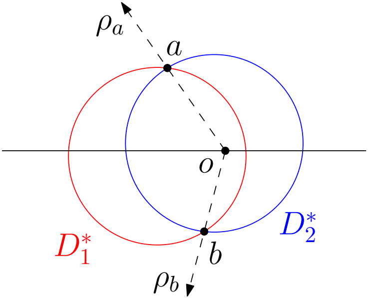

- The points to the left (resp., right) of ρ_a & ρ_b are in D_1^* (resp., D_2^*).

The nearby case

- For $i \in [n']$ and $j \in [n'']$, consider the following problem: Finding a pair of congruent disks (D_1, D_2) of smallest radus which bichromatically covers S such that $L_{i,j} \subseteq D_1$ and $R_{i,j} \subseteq D_2$.

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- Let $r^*_{i,j}$ be the optimal radii for the above problem.
- We have $r^* \leq r^*_{i,j}$ for all $i \in [n']$ and $j \in [n'']$.
- We have $r^* = r^*_{i,j}$ for some $i \in [n']$ and $j \in [n'']$.
- Therefore, $r^* = \min_{i,j} r^*_{i,j}$.

The nearby case

Lemma

Given $i \in [n']$ and $j \in [n'']$, $r^*_{i,j}$ can be computed in $O(n \log^2 n)$ time.

- **Proof idea:** parametric search + decision

The nearby case

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$r^*_{1,1}$	$r^*_{1,2}$	$\cdots$	$r^*_{1,n''}$
$r^*_{2,1}$	$r^*_{2,2}$	$\cdots$	$r^*_{2,n''}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$r^*_{n',1}$	$r^*_{n',2}$	$\cdots$	$r^*_{n',n''}$

M

- Each entry of M can be computed in $O(n \log^2 n)$ time.
- Want the smallest entry of the matrix M.

The nearby case

- **A naïve way to compute r^***
  Evaluating all $\Theta(n^2)$ entries of the matrix M: $\Theta(n^3 \log^2 n)$ time.

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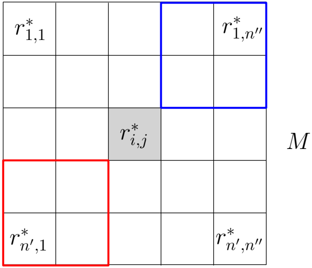
- **A naïve way to compute r^***
  Evaluating all $\Theta(n^2)$ entries of the matrix M: $\Theta(n^3 \log^2 n)$ time.
- **A better way: $O(n^2 \log^2 n)$ time**
  Apply matrix search technique: only evaluating $O(n)$ entries of M, we can obtain r^*.
  Main idea: After considering a subproblem on an entry, either its upperright or lowerleft submatrix can be pruned.



Putting everything together

- r_1^* = the "r^*" returned by the distant-case algorithm.
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Putting everything together

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- r_2^* = the "r^*" returned by the nearby-case algorithm.
- $r^* = \min\{r_1^*, r_2^*\}$.

Theorem

There is an exact algorithm for the plane bichromatic 2-center problem using $O(n^2 \log^2 n)$ time, where n is the input size.

Future work

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- Further improve the algorithms?

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- Higher dimensions or more general settings?

Related work

- L_∞ case: $O(n)$ time [Arkin et al., 2015]

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- L_∞ case: $O(n)$ time [Arkin et al., 2015]

- Min-Sum: minimizing the sum of the radii of the red and blue disks
  - Euclidean case: $O(n^4 \log^2 n)$ time [Arkin et al., 2015]
  - L_∞ case: $O(n \log^2 n)$ deterministic time or $O(n \log n)$ expected time [Arkin et al., 2015]

Thank you!
Q & A