Near-optimal Algorithms for Shortest Paths in Weighted Unit-Disk Graphs

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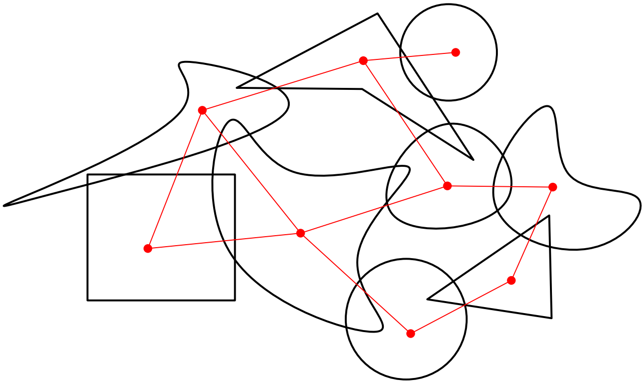
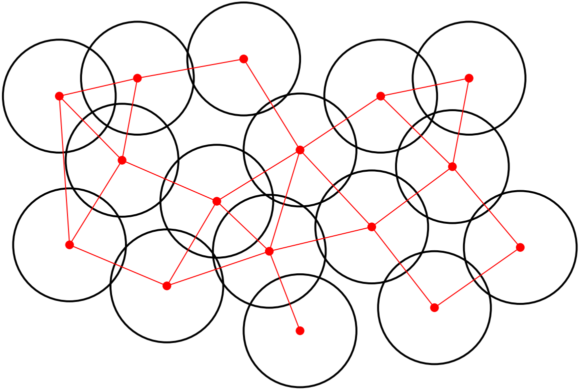
Student Presentation

Background

- **Geometric intersection graphs**
  The intersection graph of a set of geometric objects



- **Unit-disk graphs (UDGs)**
  The intersection graph of unit-disks or disks of identical radii



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- SSSP on UDGs?
- **Two ways to weight a UDG**
  1. Edges are weighted identically (Unweighted UDGs)
  2. Edges are weighted using Euclidean distances between the disk centers (Weighted UDGs)

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  - Johnson's algorithm
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- The input size for an n-vertex UDG is only $O(n)$.
- However, $|E| = \Omega(n^2)$ in an n-vertex UDG in worst case.

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- The input size for an n-vertex UDG is only $O(n)$.
- However, $|E| = \Omega(n^2)$ in an n-vertex UDG in worst case.
- Maybe we can break the $\Omega(|E|)$ lower bound for UDGs?

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- **SSSP on weighted UDGs**
  - $O(n^{1+\delta})$ time and $O(n^{1+\delta})$ space for any $\delta > 0$ by [Cabello and Jejčič 2015]
  - $O(n \log^{12+o(1)} n)$ expected time and $O(n \log^3 n)$ space (randomized) by [Kaplan et al. 2017]
  - $O(n \log n/\varepsilon^2)$ time and $O(n/\varepsilon^2)$ space for $(1+\varepsilon)$-approximation by [Chan and Skrepetos 2016]

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Theorem (Approximation algorithm)

There is a $(1+\varepsilon)$-approximate SSSP algorithm on weighted UDGs using $O(n \log n + n \log^2(1/\varepsilon))$ time and $O(n)$ space, where n is the input size.

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- **The OIWNN problem in $\mathbb{R}^2$**
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  [Goal] answer all queries
- We reduce SSSP on weighted UDGs to the OIWNN problem in $\mathbb{R}^2$.

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Theorem (Exact)

If the OIWNN problem with n operations can be solved in $f(n)$ time, then SSSP on weighted UDGs can be solved in $O(n \log n + f(n))$ time.

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- Two points $a, b \in S$ are connected by an edge iff $\|a - b\| \leq 1$. (The edge is weighted by $\|a - b\|$.)
- Let $s \in S$ be a given source.
- Our goal is to compute a table dist$[\cdot]$, where dist$[a]$ stores the length of the shortest path from s to a, for all $a \in S$.

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- The previous works [Cabello and Jejčič 2015] and [Kaplan et al. 2017] use Dijkstra's algorithm + dynamic bichromatic closest pair
- The framework of our SSSP algorithm is different from Dijkstra's, but it exploits the basic intuition of Dijkstra's.

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- The table dist$'$ is used for lazy update in case $U \cap V \neq \emptyset$.

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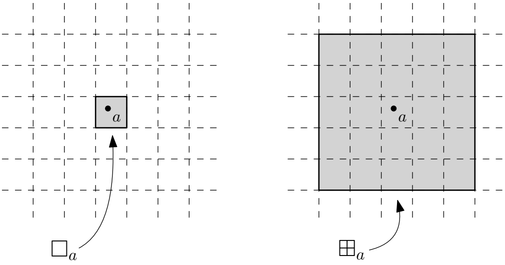

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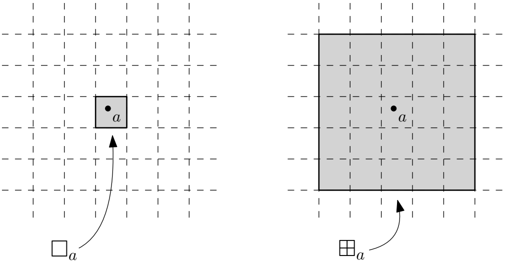

- All points in $S \cap \square_a$ are neighbors of a.
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- **Our SSSP algorithm**
  1. dist[s] $\leftarrow 0$, dist[a] $\leftarrow \infty$ for all $a \in V \backslash \{s\}$, $A \leftarrow V$
  2. Pick $c \in A$ with the smallest dist[c]
  3. $\textsc{Update}(A \cap \boxplus_c, A \cap \square_c)$
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- If we forget **Step 3**, the main difference between Dijkstra's and ours is
  - Dijkstra's considers the single point c in each iteration.
  - Ours considers all points in $\square_c$ in each iteration.

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- But we need the correctness to do **Step 4** and **Step 5** safely.
- **Step 3** is used to make dist[a] correct for all $a \in A \cap \Box_c$.

Lemma

After **Step 3** *of our algorithm,* dist[a] *equals to the length of the shortest path from s to a for all $a \in A \cap \Box_c$.*

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- What if we remove the constraint that u_v is a neighbor of v?
  Then u_v is exactly the weighted nearest neighbor of v in $A \cap \boxplus_c$ (each $u \in A \cap \boxplus_c$ is assigned the weight $\text{dist}'[u]$).
  In this case, the problem can be solved by building a WVD on $A \cap \boxplus_c$.

The exact SSSP algorithm

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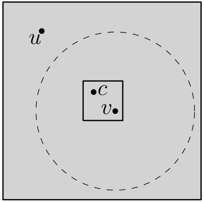

- $\text{dist}'[u] \geq \text{dist}'[c]$
- $\|u - v\| > 1 \geq \|c - v\|$
- $\implies \text{dist}'[u] + \|u - v\| > \text{dist}'[c] + \|c - v\| \implies u \neq u_v$

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- Basic idea: reducing to the OIWNN problem
  Step 4 can be done in $O(m \log m + f(m))$ time where $m = |A \cap \boxplus_c|$ and $f(m)$ is the time for solving an m-operation OIWNN instance.

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  Step 4 can be done in $O(m \log m + f(m))$ time where $m = |A \cap \boxplus_c|$ and $f(m)$ is the time for solving an m-operation OIWNN instance.
- By showing $f(m) = O(m \log^2 m)$, we conclude the following.

Theorem

There is an SSSP algorithm on weighted UDGs using $O(n \log^2 n)$ time and $O(n)$ space, where n is the input size.

Open questions

- Improve the running time to $O(n \log n)$?
- APSP in weighted UDGs in $o(n \log^2 n)$ time?
- Can our approach be used to solve other problems in UDGs?

Thank you!

Q & A