CS 6300 | **Probability Practice** | **February 1, 2022**

1. Consider the joint distribution $P(X, Y)$ below.

X	Y	$P(X,Y)$
$+x$	$+y$	0.2
$+x$	$-y$	0.3
$-x$	$+y$	0.4
$-x$	$-y$	0.1

Events

(a) $P(+x, +y) = 0.2$.

(b) $P(+x) = 0.2 + 0.3 = 0.5$.

(c) $P(-y \vee +x) = 0.2 + 0.3 + 0.1 = 0.6 = 1 - P(-x, +y) = 1 - 0.4$.

Marginal Distributions Find $P(X)$ and $P(Y)$.

X	$P(X)$
$+x$	$0.2 + 0.3 = 0.5$
$-x$	$0.4 + 0.1 = 0.5$

Y	$P(Y)$
$+y$	$0.2 + 0.4 = 0.6$
$-y$	$0.3 + 0.1 = 0.4$

Conditional Probabilities

(a) $P(+x| + y) = P(+x, +y)/P(+y) = 0.2/0.6 = 1/3$.

(b) $P(-x| + y) = P(-x, +y)/P(+y) = 0.4/0.6 = 2/3$.

(c) $P(-y| + x) = P(+x, -y)/P(+x) = 0.3/0.5 = 3/5$.

Normalization Trick $P(X| - y) = \alpha P(X, -y)$, where $\alpha = 1/(0.3 + 0.1) = 1/0.4$ from the table below.

X	$-y$	$P(X,-y)$
$+x$	$-y$	0.3
$-x$	$-y$	0.1

Hence

X	$-y$	$P(X\|-y)$
$+x$	$-y$	$3/4$
$-x$	$-y$	$1/4$

2. **Bayes' Rule.** Consider the probability distributions below. What is $P(W|dry)$?

X	$P(W)$
sun	0.8
$rain$	0.2

D	W	$P(D\|W)$
wet	sun	0.1
dry	sun	0.9
wet	$rain$	0.7
dry	$rain$	0.3

$$
\begin{aligned}
P(sun|dry) &= P(dry|sun)P(sun)/P(dry) \\
&= \alpha 0.9 * 0.8 = \alpha 0.72 \\
P(rain|dry) &= P(dry|rain)P(rain)/P(dry) \\
&= \alpha 0.3 * 0.2 = \alpha 0.06
\end{aligned}
$$

where $\alpha = 1/(0.72 + 0.06) = 1/0.78$. Hence

D	W	$P(W\|dry)$
dry	sun	$0.72/0.78$
dry	$rain$	$0.06/0.78$

3. Marijuana legalization has been in the news, and one of the states is having a gubernatorial election. The Libertarian candidate (random variable L) is more likely to legalize marijuana (random variable M) than the other candidates, but legalization may happen if any candidate is elected. The probabilities are modeled below.

	$+l$	$-l$
$P(L)$	0.1	0.9

Libertarian governor elected

	$P(+m\|L)$	$P(-m\|L)$
$+l$	0.667	0.333
$-l$	0.25	0.75

Marijuana legalized

(a) What is $P(+m)$?

$$\begin{aligned} P(+m) &= P(+m,+l) + P(+m,-l) \\ &= P(+m|+l)P(+l) + P(+m|-l)P(-l) \\ &= \frac{2}{3}\cdot\frac{1}{10} + \frac{1}{4}\cdot\frac{9}{10} \\ &= \frac{7}{24} \end{aligned}$$

(b) What is $P(+l|+m)$?

$$P(+l|+m) = \frac{P(+l,+m)}{P(+m)} = \frac{P(+m|+l)P(+l)}{P(+m)} = \frac{\frac{2}{3}\cdot\frac{1}{10}}{\frac{7}{24}} = \frac{8}{35}$$

(c) Fill in the joint distribution table below.

L	M	$P(L,M)$
$+l$	$+m$	1/15
$+l$	$-m$	1/30
$-l$	$+m$	9/40
$-l$	$-m$	27/40

Sample calculation:

$$P(+l,+m) = P(+l|+m)\cdot P(+m) = \frac{8}{35}\cdot\frac{7}{24} = \frac{1}{15}$$

(d) More information is provided with new random variables B (balanced budget) and A (workplace absenteeism).

	$P(+b\|M)$	$P(-b\|M)$
$+m$	0.4	0.6
$-m$	0.2	0.8

Balanced Budget

	$P(+a\|M)$	$P(-a\|M)$
$+m$	0.75	0.25
$-m$	0.5	0.5

Absenteeism

Fill in the joint distribution table below.

L	M	B	A	$P(L,M,B,A)$
$+l$	$+m$	$+b$	$+a$	1/50
$+l$	$+m$	$+b$	$-a$	1/150
$+l$	$+m$	$-b$	$+a$	3/100
$+l$	$+m$	$-b$	$-a$	1/100
$+l$	$-m$	$+b$	$+a$	1/300
$+l$	$-m$	$+b$	$-a$	1/300
$+l$	$-m$	$-b$	$+a$	1/75
$+l$	$-m$	$-b$	$-a$	1/75

L	M	B	A	$P(L,M,B,A)$
$-l$	$+m$	$+b$	$+a$	27/400
$-l$	$+m$	$+b$	$-a$	9/400
$-l$	$+m$	$-b$	$+a$	81/800
$-l$	$+m$	$-b$	$-a$	27/800
$-l$	$-m$	$+b$	$+a$	27/400
$-l$	$-m$	$+b$	$-a$	27/400
$-l$	$-m$	$-b$	$+a$	27/100
$-l$	$-m$	$-b$	$-a$	27/100

Sample calculation assumes independence:

$$
\begin{aligned}
P(+l,+m,+b,+a) &= P(+a,+b,+m,+l) \\
&= P(+a|+b,+m,+l)P(+b|+m,+l)P(+m|+l)P(+l) \\
&= P(+a|+m)\cdot P(+b|+m)\cdot P(+m|+l)\cdot P(+l) \\
&= \frac{3}{4}\cdot\frac{4}{10}\cdot\frac{2}{3}\cdot\frac{1}{10} = \frac{1}{50}
\end{aligned}
$$

(e) Compute the following.

i. $P(+b|+m)$
0.4 (directly from the conditional table)

ii. $P(+b|+m,+l)$
0.4 (also directly from the conditional table because of independence of B and L given M)

iii. $P(+b)$

$$
\begin{aligned}
P(+b) &= P(+b|+m)P(+m) + P(+b|-m)\cdot P(-m) \\
&= \frac{4}{10}\cdot\frac{7}{24} + \frac{1}{5}\cdot\frac{17}{24} = \frac{31}{120}
\end{aligned}
$$

iv. $P(+a|+b)$

Use the product rule. $P(+b)$ has been calculated above, and $P(+a,+b)$ is obtained by summing lines where these appear in the table above.

$$\begin{aligned} P(+a|+b) &= \frac{P(+a,+b)}{P(+b)} \\ &= \frac{\frac{1}{50}+\frac{2}{150}+\frac{27}{400}+\frac{27}{400}}{\frac{31}{120}} \\ &= \frac{19}{31} \end{aligned}$$