CS 6300: Artificial Intelligence

Decision Networks and Value of Perfect Information

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[Based on slides created by Dan Klein and Pieter Abbeel http://ai.berkeley.edu]
Decision Networks

- **Maximize Expected Utility**
  - choose the action which maximizes the expected utility given the evidence
- Can directly operationalize this with decision networks
  - Bayes nets with nodes for utility and actions
  - Lets us calculate the expected utility for each action

- New node types:
  - Chance nodes (just like BNs)
  - Actions (rectangles, cannot have parents, act as observed evidence)
  - Utility node (diamond, depends on action and chance nodes)
Decision Networks

- **Action selection**
  - Instantiate all evidence
  - Set action node(s) each possible way
  - Calculate posterior for all parents of utility node, given the evidence
  - Calculate expected utility for each action
  - Choose maximizing action

![Decision Network Diagram](image-url)
Decision Networks

Umbrella = leave

\[ EU(\text{leave}) = \sum_w P(w)U(\text{leave}, w) = 0.7 \cdot 100 + 0.3 \cdot 0 = 70 \]

Umbrella = take

\[ EU(\text{take}) = \sum_w P(w)U(\text{take}, w) = 0.7 \cdot 20 + 0.3 \cdot 70 = 35 \]

Optimal decision = leave

\[ \text{MEU}(\varnothing) = \max_a EU(a) = 70 \]

<table>
<thead>
<tr>
<th>A</th>
<th>W</th>
<th>U(A,W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>leave</td>
<td>sun</td>
<td>100</td>
</tr>
<tr>
<td>leave</td>
<td>rain</td>
<td>0</td>
</tr>
<tr>
<td>take</td>
<td>sun</td>
<td>20</td>
</tr>
<tr>
<td>take</td>
<td>rain</td>
<td>70</td>
</tr>
</tbody>
</table>
Example: Decision Networks

Umbrella = leave

\[ EU(\text{leave}|\text{bad}) = \sum_w P(w|\text{bad})U(\text{leave}, w) \]

\[ = 0.34 \cdot 100 + 0.66 \cdot 0 = 34 \]

Umbrella = take

\[ EU(\text{take}|\text{bad}) = \sum_w P(w|\text{bad})U(\text{take}, w) \]

\[ = 0.34 \cdot 20 + 0.66 \cdot 70 = 53 \]

Optimal decision = take

\[ \text{MEU}(F = \text{bad}) = \max_a EU(a|\text{bad}) = 53 \]

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</tr>
<tr>
<td>take</td>
<td>rain</td>
<td>70</td>
</tr>
</tbody>
</table>

| W     | P(W|F=bad) |
|-------|------------|
| sun   | 0.34       |
| rain  | 0.66       |
Decisions as Outcome Trees

Decision diagram tells us what numbers to put in our expectimax tree!
Value of Information
Value of Perfect Information (VPI)

- Idea: compute value of acquiring evidence
  - Can be done directly from decision network

- Example: buying oil drilling rights
  - Two blocks A and B, exactly one has oil, worth k
  - You can drill in one location
  - Prior probabilities 0.5 each, & mutually exclusive
  - Drilling in either A or B has EU = k/2, MEU = k/2

- Question: what’s the value of information of O?
  - Value of knowing which of A or B has oil
  - Value is expected gain in MEU from new info
  - Survey may say “oil in a” or “oil in b,” prob 0.5 each
  - If we know OilLoc, MEU is k (either way)
  - Gain in MEU from knowing OilLoc?
  - VPI(OilLoc) = k/2
  - Fair price of information: k/2
VPI Example: Weather

MEU with no evidence

\[ \text{MEU}(\emptyset) = \max_a \text{EU}(a) = 70 \]

MEU if forecast is bad

\[ \text{MEU}(F = \text{bad}) = \max_a \text{EU}(a|\text{bad}) = 53 \]

MEU if forecast is good

\[ \text{MEU}(F = \text{good}) = \max_a \text{EU}(a|\text{good}) = 95 \]

Forecast distribution

<table>
<thead>
<tr>
<th>F</th>
<th>P(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>good</td>
<td>0.59</td>
</tr>
<tr>
<td>bad</td>
<td>0.41</td>
</tr>
</tbody>
</table>

\[ \text{VPI}(E'|e) = \left( \sum_{e'} P(e'|e) \text{MEU}(e, e') \right) - \text{MEU}(e) \]

\[ 0.59 \cdot (95) + 0.41 \cdot (53) - 70 = 77.8 - 70 = 7.8 \]
Value of Information

- Assume we have evidence $E=e$. Value if we act now:
  \[
  \text{MEU}(e) = \max_a \sum_s P(s|e) \ U(s, a)
  \]

- Assume we see that $E' = e'$. Value if we act then:
  \[
  \text{MEU}(e, e') = \max_a \sum_s P(s|e, e') \ U(s, a)
  \]

- BUT $E'$ is a random variable whose value is unknown, so we don’t know what $e'$ will be

- Expected value if $E'$ is revealed and then we act:
  \[
  \text{MEU}(e, E') = \sum_{e'} P(e'|e) \text{MEU}(e, e')
  \]

- Value of information: how much MEU goes up by revealing $E'$ first then acting, over acting now:
  \[
  \text{VPI}(E'|e) = \text{MEU}(e, E') - \text{MEU}(e)
  \]
**VPI Properties**

- **Nonnegative**
  \[
  \forall E', e : \text{VPI}(E'|e) \geq 0
  \]

- **Nonadditive**
  (think of observing \(E_j\) twice)
  \[
  \text{VPI}(E_j, E_k|e) \neq \text{VPI}(E_j|e) + \text{VPI}(E_k|e)
  \]

- **Order-independent**
  \[
  \text{VPI}(E_j, E_k|e) = \text{VPI}(E_j|e) + \text{VPI}(E_k|e, E_j)
  \]
  \[
  = \text{VPI}(E_k|e) + \text{VPI}(E_j|e, E_k)
  \]
Quick VPI Questions

- The soup of the day is either clam chowder or split pea, but you wouldn’t order either one. What’s the value of knowing which it is?

- There are two kinds of plastic forks at a picnic. One kind is slightly sturdier. What’s the value of knowing which?

- You’re playing the lottery. The prize will be $0 or $100. You can play any number between 1 and 100 (chance of winning is 1%). What is the value of knowing the winning number?
Value of Imperfect Information?

- No such thing
- Information corresponds to the observation of a node in the decision network
- If data is “noisy” that just means we don’t observe the original variable, but another variable which is a noisy version of the original one
VPI Question

- $\text{VPI(OilLoc)} \ g \ k/2$
- $\text{VPI(ScoutingReport)} \ g \ 0$
- $\text{VPI(Scout)} \ w \ 0$
- $\text{VPI(Scout | ScoutingReport)} \ g \ 0$

Generally:
If $\text{Parents(U) \perp \!\!\!\!\perp Z | CurrentEvidence}$
Then $\text{VPI(Z | CurrentEvidence)} = 0$
Markov Models: Bayes’ Nets + Time
Reasoning over Time or Space

- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring

- Need to introduce time (or space) into our models
Markov Models

- Value of X at a given time is called the state

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

\[ P(X_1) \quad P(X_t|X_{t-1}) \]
Joint Distribution of a Markov Model

- Joint distribution:
  \[ P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3) \]

- More generally:
  \[ P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2) \ldots P(X_T|X_{T-1}) \]
  \[ = P(X_1) \prod_{t=2}^{T} P(X_t|X_{t-1}) \]

- Questions:
  - Does this indeed define a joint distribution?
  - Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?
Implied Conditional Independencies

- Bayes’ net implies \( X_3 \perp X_1 \mid X_2 \) and \( X_4 \perp X_1, X_2 \mid X_3 \)

- Do we also have \( X_1 \perp X_3, X_4 \mid X_2 \)?
  - Yes!
  - D-Separation
Markov Models Recap

- Explicit assumption for all $t$:
  $$X_t \perp X_1, \ldots, X_{t-2} \mid X_{t-1}$$

- Consequence, joint distribution can be written as:
  $$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$
  $$= P(X_1) \prod_{t=2}^{T} P(X_t|X_{t-1})$$

- Implied conditional independencies:
  - Past variables independent of future variables given the present
    i.e., if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$ then: $X_{t_1} \perp X_{t_3} \mid X_{t_2}$

- Additional explicit assumption: $P(X_t \mid X_{t-1})$ is the same for all $t$
Example Markov Chain: Weather

- **States:** $X = \{\text{rain, sun}\}$

- **Initial distribution:** 1.0 sun

- **CPT $P(X_t \mid X_{t-1})$:**

<table>
<thead>
<tr>
<th>$X_{t-1}$</th>
<th>$X_t$</th>
<th>$P(X_t \mid X_{t-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sun</td>
<td>sun</td>
<td>0.9</td>
</tr>
<tr>
<td>sun</td>
<td>rain</td>
<td>0.1</td>
</tr>
<tr>
<td>rain</td>
<td>sun</td>
<td>0.3</td>
</tr>
<tr>
<td>rain</td>
<td>rain</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Two new ways of representing the same CPT.
Example Markov Chain: Weather

- **Initial distribution:** $1.0 \text{ sun}

- What is the probability distribution after one step?

\[
P(X_2 = \text{sun}) = \sum_{X_1} P(X_2 = \text{sun} | X_1) P(X_1)
\]

\[
P(X_2 = \text{sun} | X_1 = \text{sun}) P(X_1 = \text{sun}) + P(X_2 = \text{sun} | X_1 = \text{rain}) P(X_1 = \text{rain})
\]

\[
0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9
\]

\[
P(X_2 = \text{rain}) = 0.1
\]
Mini-Forward Algorithm

- Question: What’s $P(X)$ on some day $t$?

\[
P(x_1) = \text{known}
\]

\[
P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)
\]

\[
= \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1})
\]

Forward simulation
### Example Run of Mini-Forward Algorithm

#### From initial observation of sun

<table>
<thead>
<tr>
<th></th>
<th>$P(X_1)$</th>
<th>$P(X_2)$</th>
<th>$P(X_3)$</th>
<th>$P(X_4)$</th>
<th>$P(X_{\infty})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9</td>
<td>0.84</td>
<td>0.804</td>
<td>0.75</td>
<td>0.25</td>
</tr>
<tr>
<td>0</td>
<td>0.1</td>
<td>0.16</td>
<td>0.196</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>

#### From initial observation of rain

<table>
<thead>
<tr>
<th></th>
<th>$P(X_1)$</th>
<th>$P(X_2)$</th>
<th>$P(X_3)$</th>
<th>$P(X_4)$</th>
<th>$P(X_{\infty})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.3</td>
<td>0.48</td>
<td>0.588</td>
<td>0.75</td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.7</td>
<td>0.52</td>
<td>0.412</td>
<td>0.25</td>
</tr>
</tbody>
</table>

#### From yet another initial distribution $P(X_1)$:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$1 - p$</th>
<th>$P(X_1)$</th>
<th>$P(X_{\infty})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.75</td>
<td>0.25</td>
</tr>
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</table>
Stationary Distributions

- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution

- Stationary distribution:
  - The distribution we end up with is called the stationary distribution $P_\infty$ of the chain
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$
Example: Stationary Distributions

- Question: What’s $P(X)$ at time $t = \infty$?

\[ P_{\infty}(\text{sun}) = P(\text{sun}|\text{sun})P_{\infty}(\text{sun}) + P(\text{sun}|\text{rain})P_{\infty}(\text{rain}) \]
\[ P_{\infty}(\text{rain}) = P(\text{rain}|\text{sun})P_{\infty}(\text{sun}) + P(\text{rain}|\text{rain})P_{\infty}(\text{rain}) \]

\[
\begin{align*}
P_{\infty}(\text{sun}) &= 0.9P_{\infty}(\text{sun}) + 0.3P_{\infty}(\text{rain}) \\
P_{\infty}(\text{rain}) &= 0.1P_{\infty}(\text{sun}) + 0.7P_{\infty}(\text{rain}) \\
P_{\infty}(\text{sun}) &= 3P_{\infty}(\text{rain}) \\
P_{\infty}(\text{rain}) &= \frac{1}{3}P_{\infty}(\text{sun})
\end{align*}
\]

Also: $P_{\infty}(\text{sun}) + P_{\infty}(\text{rain}) = 1$

\[
\begin{align*}
P_{\infty}(\text{sun}) &= \frac{3}{4} \\
P_{\infty}(\text{rain}) &= \frac{1}{4}
\end{align*}
\]
Application of Stationary Distribution: Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. $c$, uniform jump to a random page (dotted lines, not all shown)
    - With prob. $1-c$, follow a random outlink (solid lines)

- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)
Next Time: Hidden Markov Models!