CS 6300: Artificial Intelligence

Bayes' Nets: Independence and Basic Inference

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[Based on slides created by Dan Klein and Pieter Abbeel http://ai.berkeley.edu.]

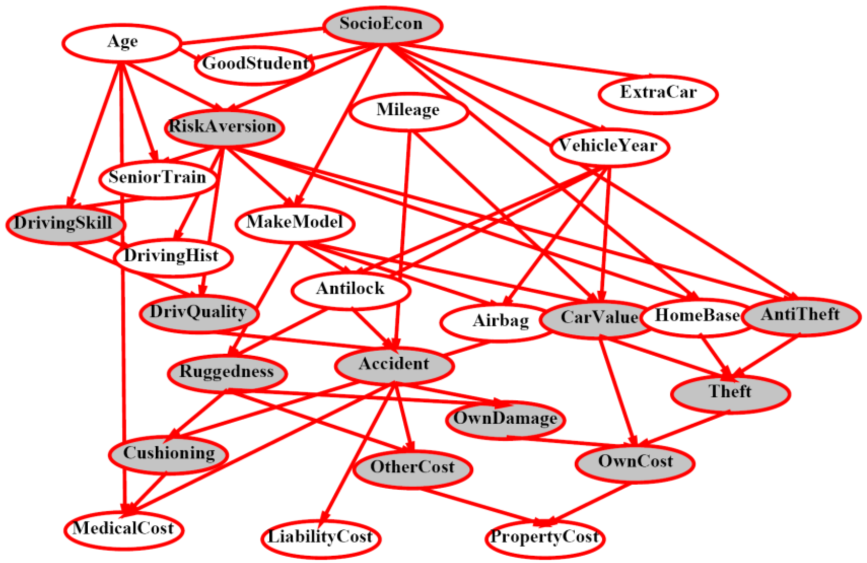
Probability Recap

- Conditional probability $$P(x|y) = \frac{P(x,y)}{P(y)}$$
- Product rule $$P(x,y) = P(x|y)P(y)$$
- Chain rule $$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1,X_2)\ldots \\ &= \prod_{i=1}^{n} P(X_i|X_1,\ldots,X_{i-1}) \end{aligned}$$
- X, Y independent if and only if: $\forall x,y : P(x,y) = P(x)P(y)$ $\quad X \perp\!\!\!\perp Y$
- X and Y are conditionally independent given Z if and only if: $X \perp\!\!\!\perp Y | Z$

$$\forall x,y,z : P(x,y|z) = P(x|z)P(y|z)$$

Bayes' Nets

- A Bayes' net is an efficient encoding of a probabilistic model of a domain



- Questions we can ask:
  - Inference: given a fixed BN, what is P(X | e)?
  - Representation: given a BN graph, what kinds of distributions can it encode?
  - Modeling: what BN is most appropriate for a given domain?

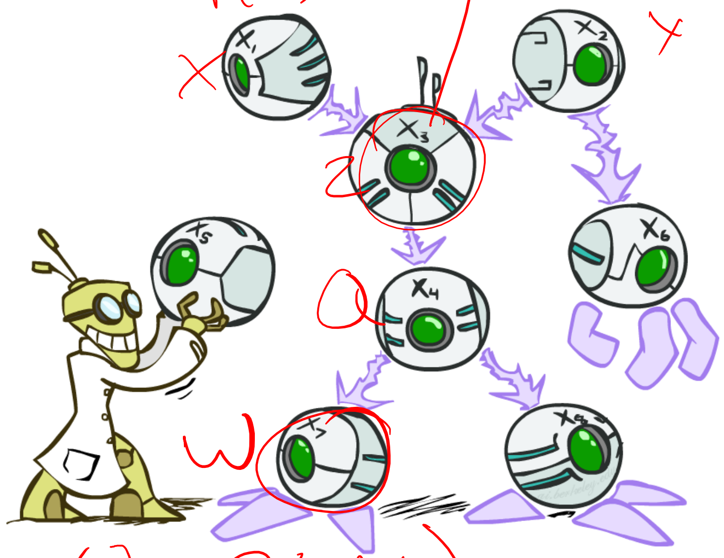
Bayes' Net Semantics

$P(Z|X,Y)$

DAG

- A directed, acyclic graph, one node per random variable
- A conditional probability table (CPT) for each node
  - A collection of distributions over X, one for each combination of parents' values

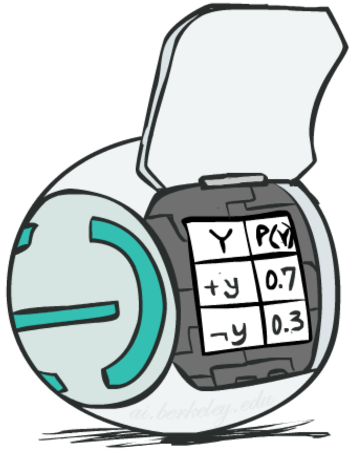
$$P(X|a_1 \ldots a_n)$$



- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

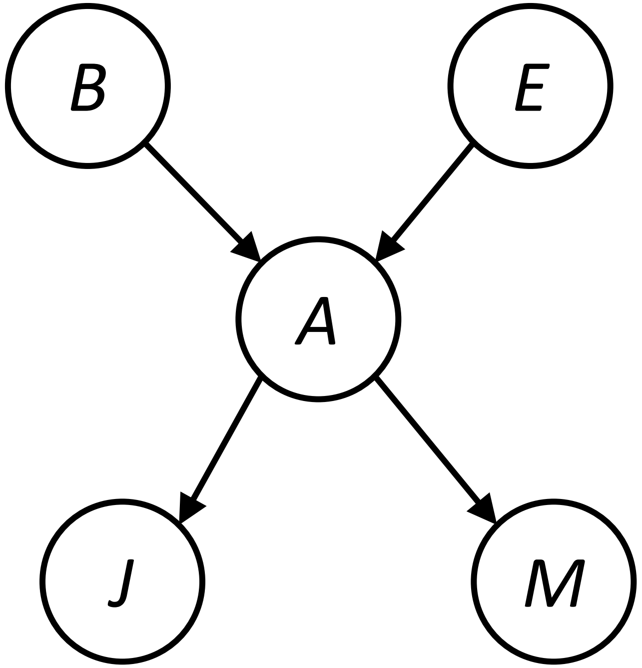
$P(W|X_1 \ldots X_n) = P(W|Q)$

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i|\textit{parents}(X_i))$$

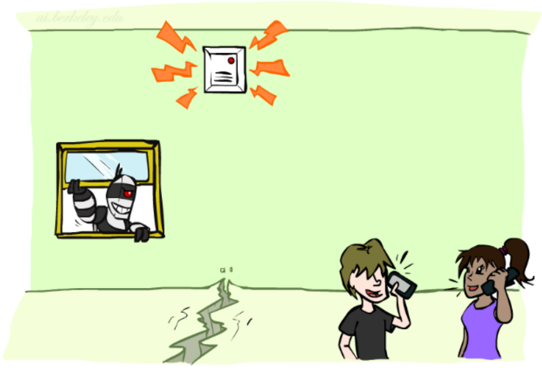


Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998



A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$$P(+b,-e,+a,-j,+m) = P(+b)P(-e)P(+a|+b,-e) \cdot P(-j|+a)P(+m|+a)$$

Example: Alarm Network

B	P(B)
+b	0.001
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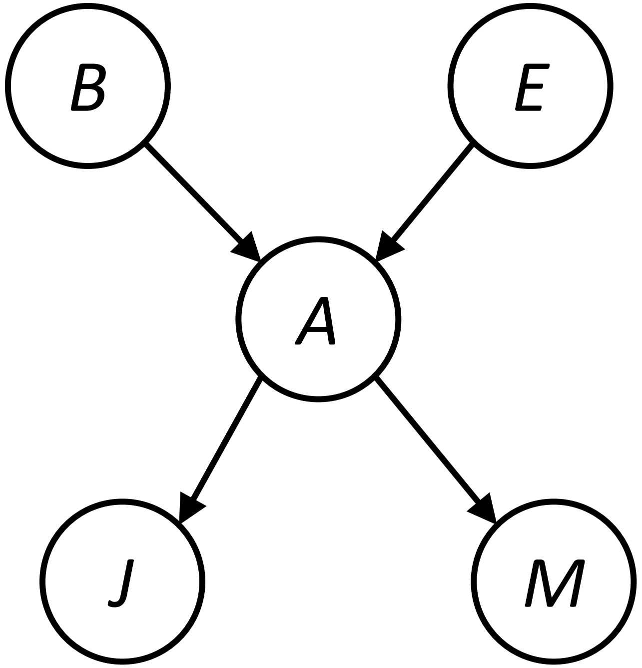

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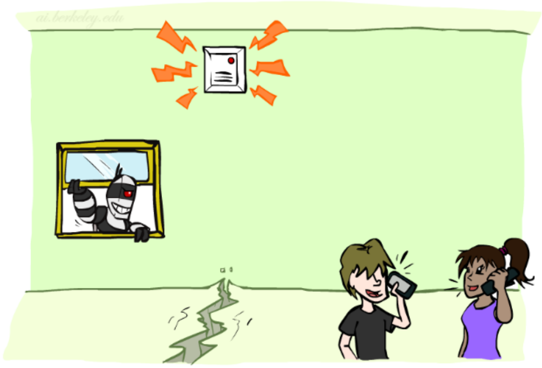

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B	E	A	P(A|B,E)
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+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$$P(+b,-e,+a,-j,+m) =$$

$$P(+b)P(-e)P(+a|+b,-e)P(-j|+a)P(+m|+a) =$$

$$0.001 \times 0.998 \times 0.94 \times 0.1 \times 0.7$$

Size of a Bayes' Net

- How big is a joint distribution over N Boolean variables?

  2^N

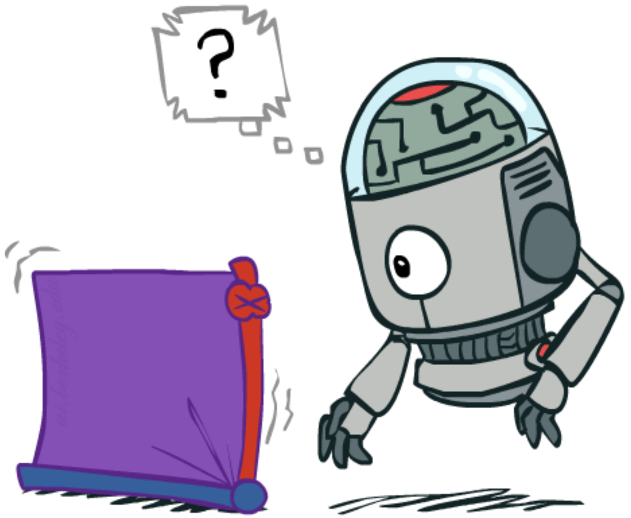
- How big is an N-node net if nodes have up to k parents?

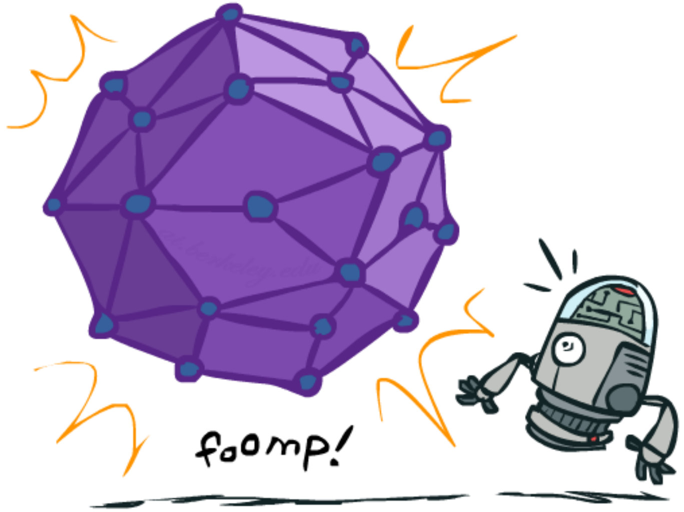
  $O(N * 2^{k+1})$

- Both give you the power to calculate

  $$P(X_1, X_2, \ldots X_n)$$

- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)





Bayes' Nets

- ✓ Representation
- Conditional Independences
- Probabilistic Inference
- Learning Bayes' Nets from Data

Conditional Independence

- X and Y are **independent** if

$$\forall x, y \;\; P(x,y) = P(x)P(y) \dashrightarrow X \perp\!\!\!\perp Y$$

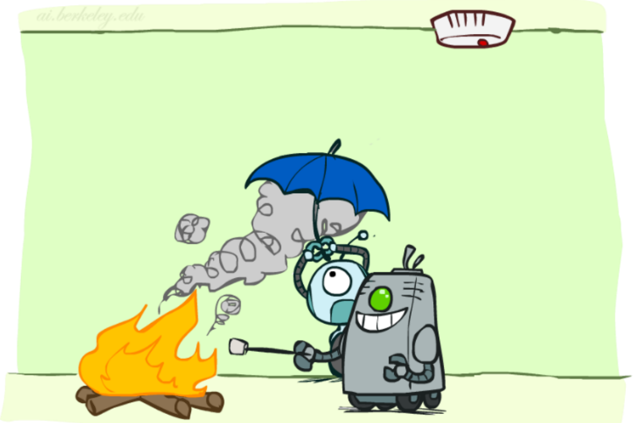
- X and Y are **conditionally independent** given Z

$P(x|y,z) = P(x|z)$

$$\forall x, y, z \;\; P(x,y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y | Z$$

- (Conditional) independence is a property of a distribution

- Example: $Alarm \perp\!\!\!\perp Fire | Smoke$

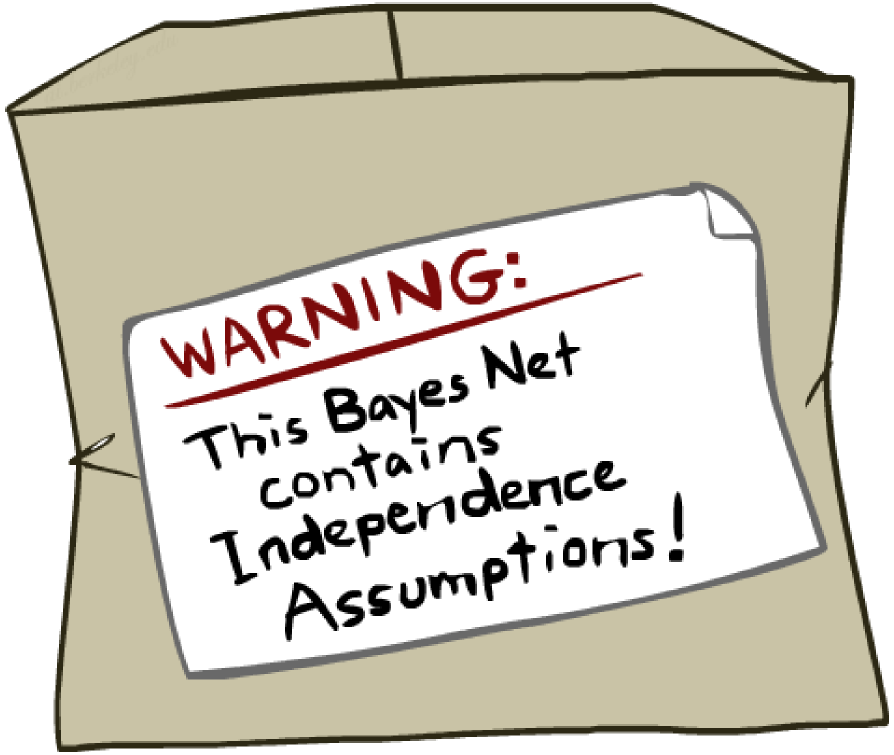


Bayes Nets: Assumptions

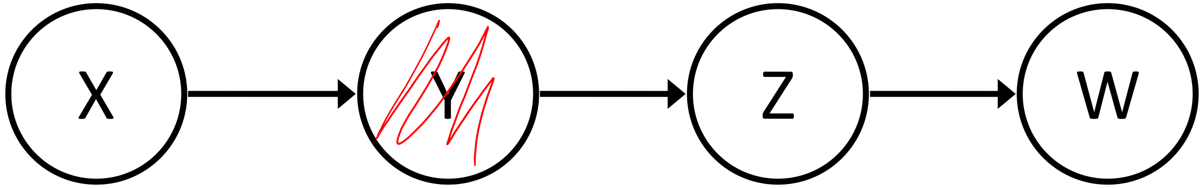
- Assumptions we are required to make to define the Bayes net when given the graph:

$$P(x_i|x_1 \cdots x_{i-1}) = P(x_i|parents(X_i))$$

- Beyond above "chain rule → Bayes net" conditional independence assumptions
  - Often additional conditional independences
  - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes net graph



Example



$$P(x,y) = P(x)P(y|x)$$

- Conditional independence assumptions directly from simplifications in chain rule:

$$P(x,y,z,w) = \underline{P(x)}\,\underline{P(y|x)}\,\underline{P(z|y)}\,\underline{P(w|z)} = \underline{P(x)}\,\underline{P(y|x)}\,P(z|x,y)\,P(\underline{w|x,y,z})$$

$$Z \perp\!\!\!\perp X \mid Y \qquad W \perp\!\!\!\perp X,Y \mid Z$$

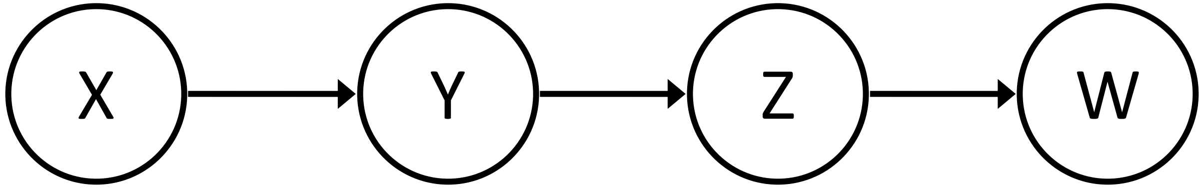
- Additional implied conditional independence assumptions?

$$\leftarrow \sum_z P(z,w|y)$$

$$X \perp\!\!\!\perp W \mid Y \qquad P(w|y,x) = P(w|y)$$

$$\frac{P(w,x,y)}{P(x,y)} = \frac{\sum_z P(x,y,z,w)}{P(x,y)} = \frac{\sum_z \cancel{P(x)}\cancel{P(y|x)}P(z|y)P(w|z)}{\cancel{P(x)}\cancel{P(y|x)}} = \sum_z P(z|y)P(w|z,y)$$

Example



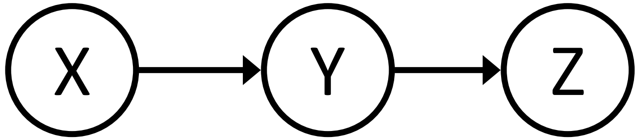
- Conditional independence assumptions directly from simplifications in chain rule:

- Additional implied conditional independence assumptions?

$$P(W|X,Y) = \frac{P(W,X,Y)}{P(X,Y)} = \frac{\sum_Z P(X)P(Y|X)P(Z|Y)P(W|Z)}{P(X)P(Y|X)} = \sum_Z P(Z|Y)P(W|Z,Y) = \sum_Z P(W,Z|Y) = P(W|Y)$$

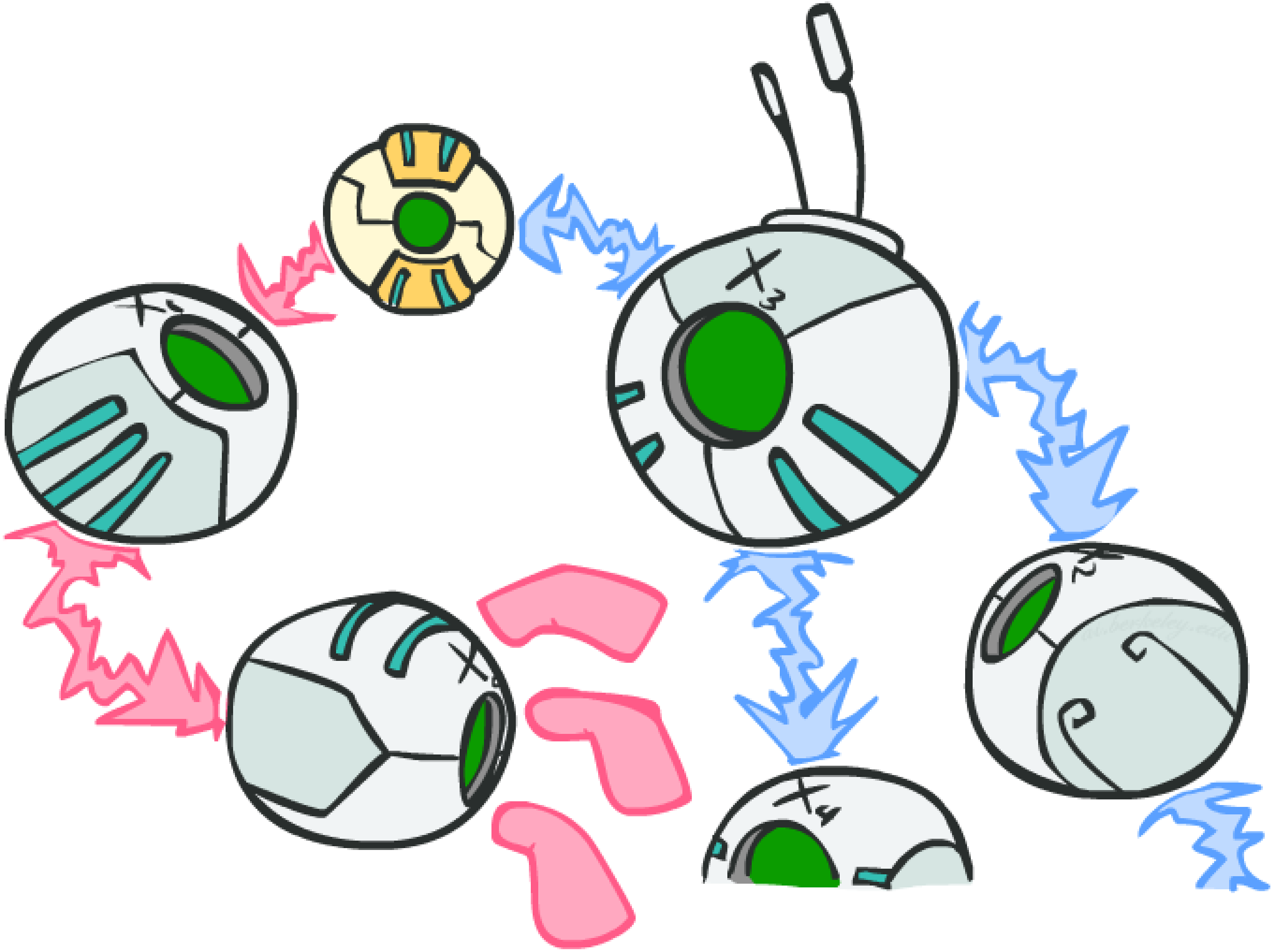
Independence in a BN

- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can prove using algebra (tedious in general)
  - If no, can prove with a counter example
  - Example:



  - Question: are X and Z necessarily independent?
    - Answer: no. Example: low pressure causes rain, which causes traffic.
    - X can influence Z, Z can influence X (via Y)
    - Addendum: they *could* be independent: how?

D-separation: Outline



D-separation: Outline

- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

Causal Chains

- This configuration is a "causal chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Example:
    - Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic
    - In numbers:

      P(+y | +x) = 1, P(-y | - x) = 1,
      P(+z | +y) = 1, P(-z | -y) = 1

Causal Chains

- This configuration is a "causal chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x,y,z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z given Y?

Baye's Net

$$P(z|x,y) = \frac{P(x,y,z)}{P(x,y)}$$

$$= \frac{\cancel{P(x)}\cancel{P(y|x)}P(z|y)}{\cancel{P(x)}\cancel{P(y|x)}}$$

$$= P(z|y)$$

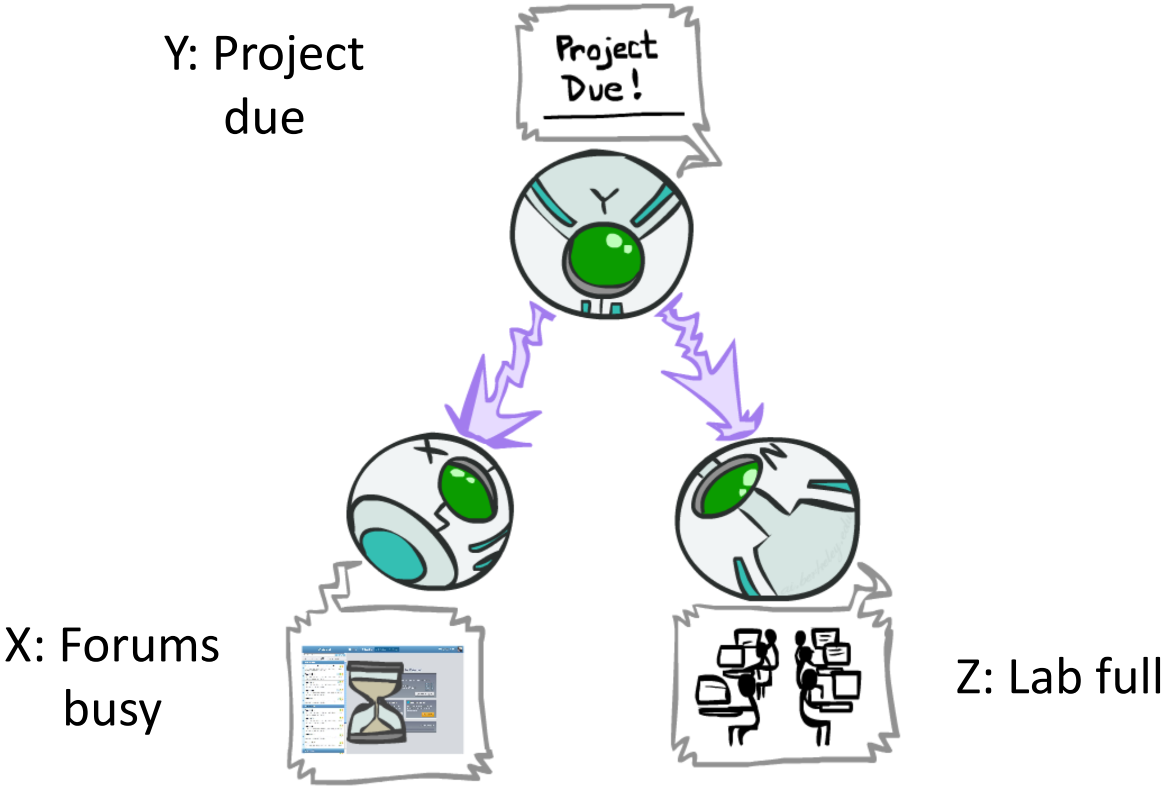
Yes!

- Evidence along the chain "blocks" the influence

Common Cause

$P(x|z) = P(x)$

- This configuration is a "common cause"



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

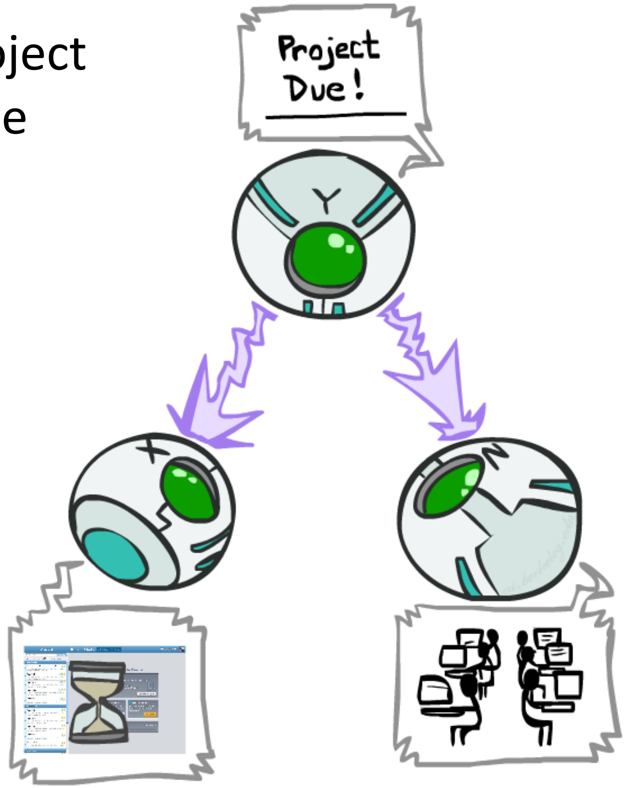
- Guaranteed X independent of Z ? *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Example:
    - Project due causes both forums busy and lab full
    - In numbers:

P(+x | +y) = 1, P(-x | -y) = 1,
P(+z | +y) = 1, P(-z | -y) = 1

Common Cause

- This configuration is a "common cause"

Y: Project due



X: Forums busy

Z: Lab full

$$P(x,y,z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X and Z independent given Y?

$$P(z|x,y) = \frac{P(x,y,z)}{P(x,y)}$$

$$= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)}$$

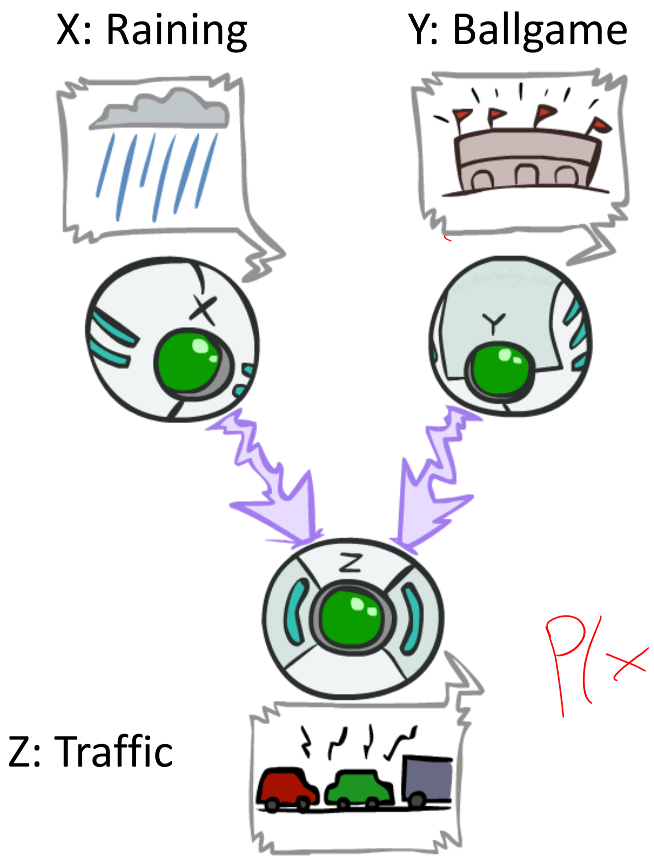
$$= P(z|y)$$

Yes!

- Observing the cause blocks influence between effects.

Common Effect

- Last configuration: two causes of one effect (v-structures)



$$X \perp\!\!\!\perp Y$$

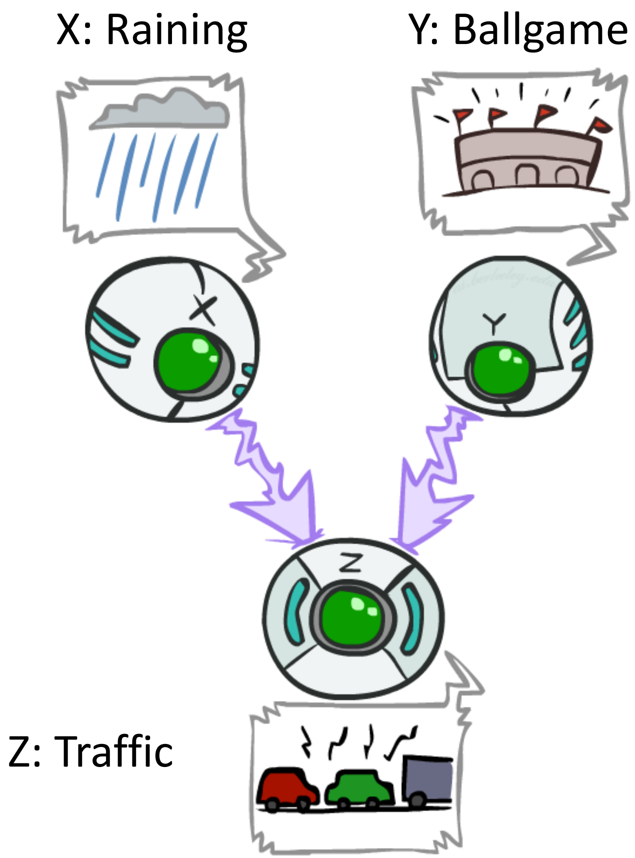
$$P(X,Y) = P(X)P(Y)$$

$$\sum_z P(x,y,z) = \sum_z P(x)P(y)P(z|x,y) = P(x)P(y)\underbrace{\cancel{\sum_z P(z|x,y)}}_{1} = P(x)P(y)$$

$$P(x,y,z) = P(x)P(y)P(z|x,y)$$

Common Effect

- Last configuration: two causes of one effect (v-structures)

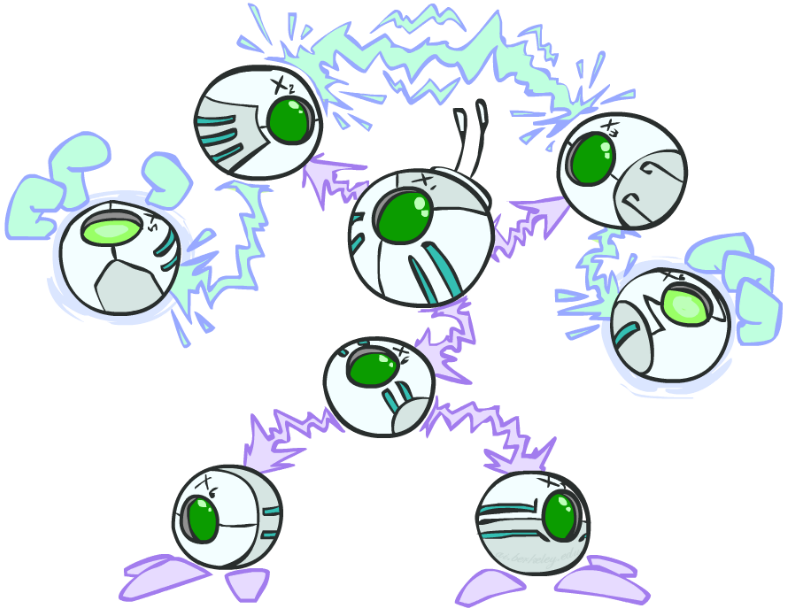


- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the other cases
  - Observing an effect activates influence between possible causes.

The General Case

The General Case

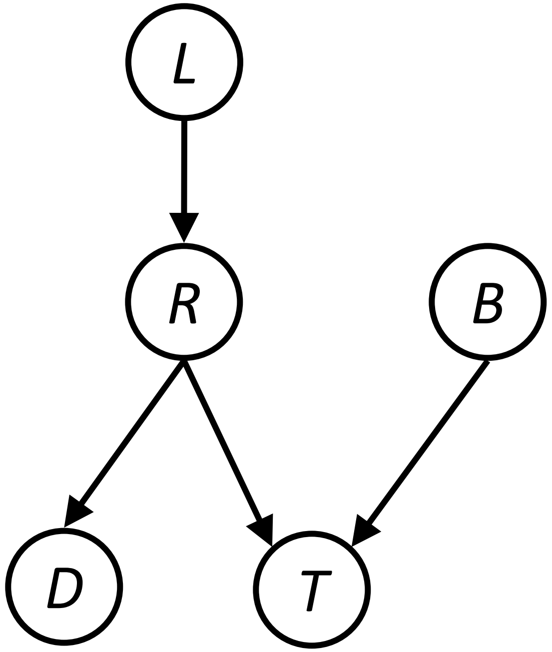
- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases

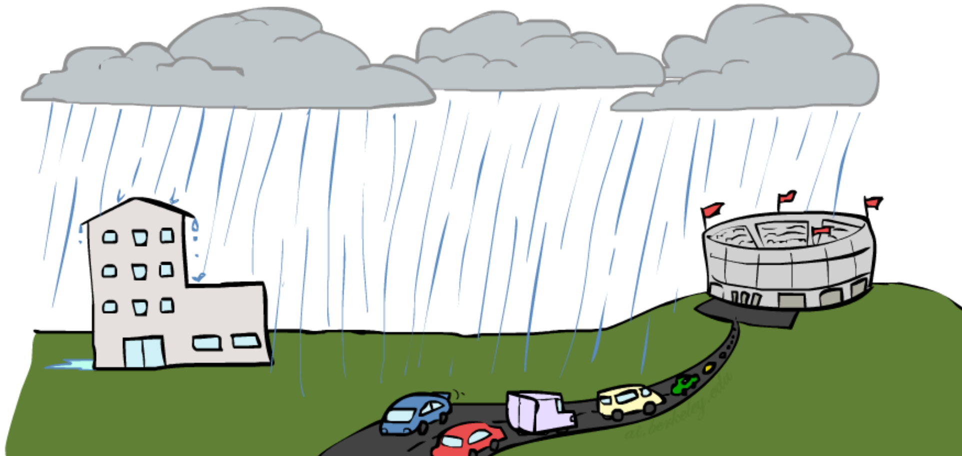


Reachability

$L \perp\!\!\!\perp B \mid E$?

- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active"



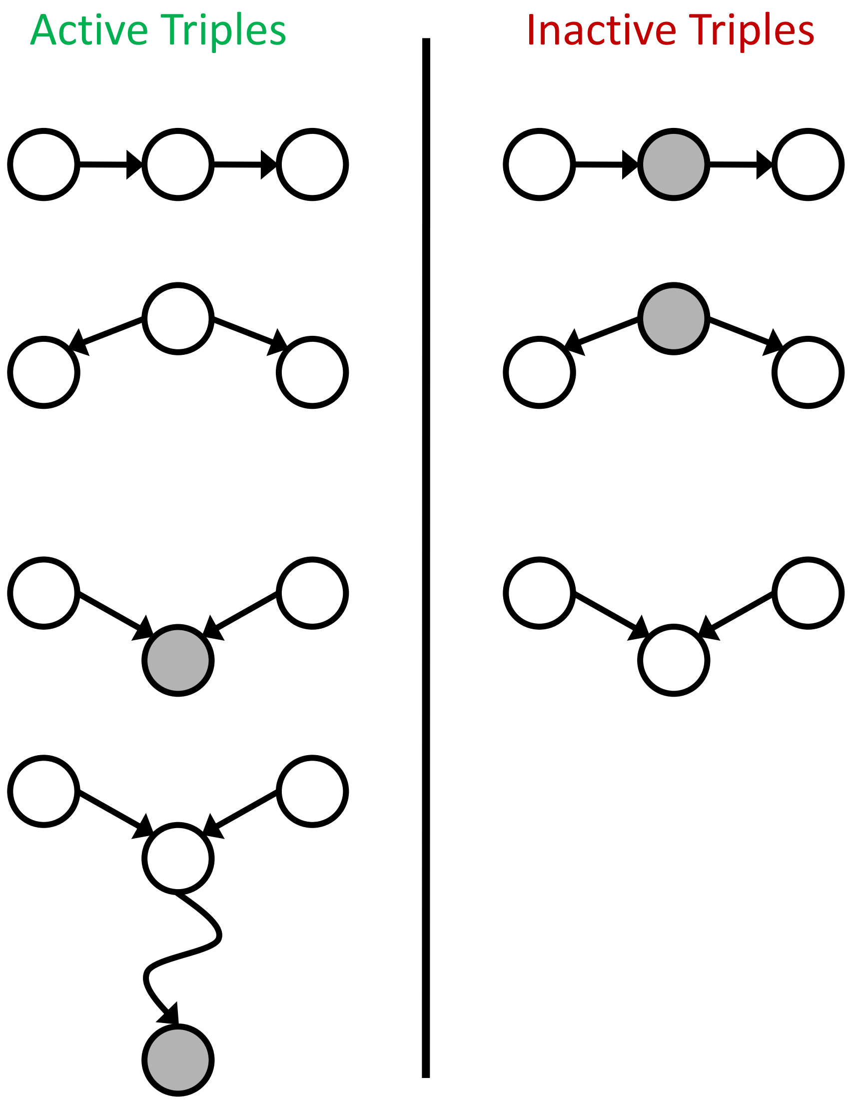


Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y "d-separated" by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = independence!

- A path is active if each triple is active:
  - Causal chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
  - Common cause $A \leftarrow B \rightarrow C$ where B is unobserved
  - Common effect (aka v-structure)
    $A \rightarrow B \leftarrow C$ where B *or one of its descendents* is observed

- All it takes to block a path is a single inactive segment



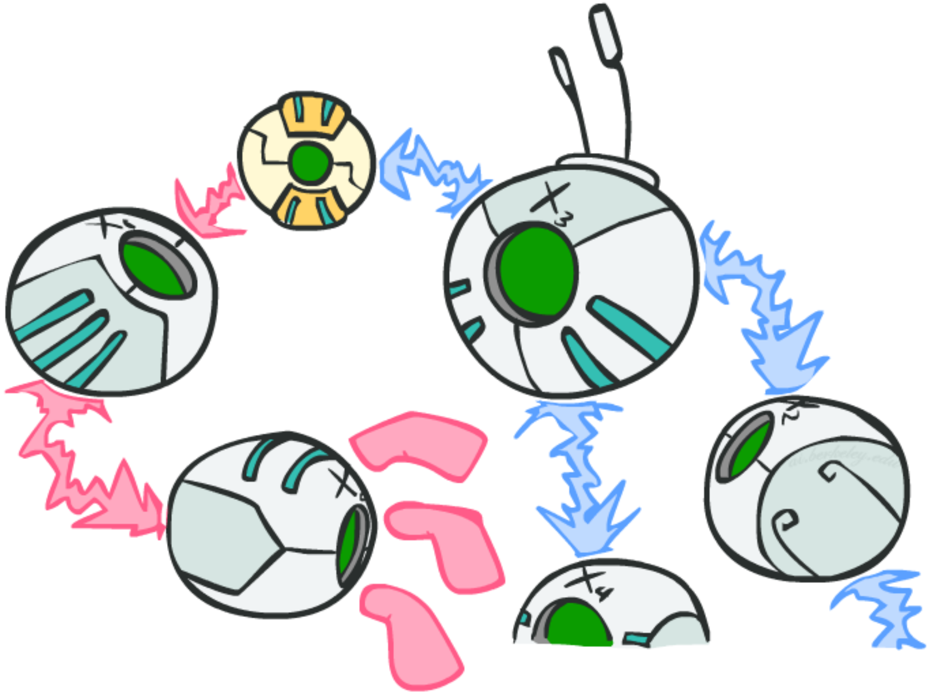
D-Separation

- Query: $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$?
- Check all (undirected!) paths between X_i and X_j
  - If one or more active, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

  - Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

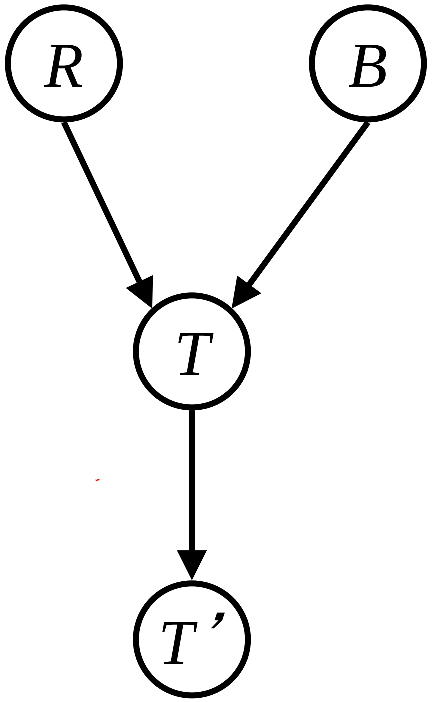


Example

$R \perp\!\!\!\perp B$ *Yes*

$R \perp\!\!\!\perp B | T$ *No*

$R \perp\!\!\!\perp B | T'$ *No*



Example

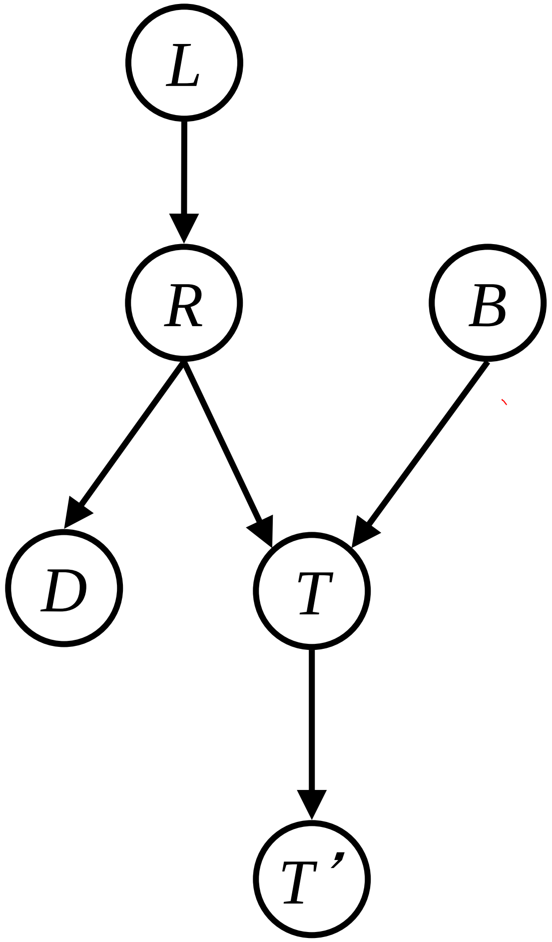
$L \perp\!\!\!\perp T' | T$ Yes

$L \perp\!\!\!\perp B$ Yes

$L \perp\!\!\!\perp B | T$ No

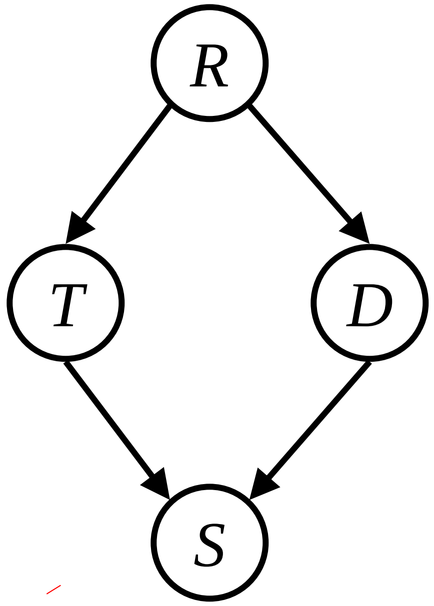
$L \perp\!\!\!\perp B | T'$ No

$L \perp\!\!\!\perp B | T, R$ Yes



Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I’m sad
- Questions:



$T \perp\!\!\!\perp D$ No

$T \perp\!\!\!\perp D | R$ Yes

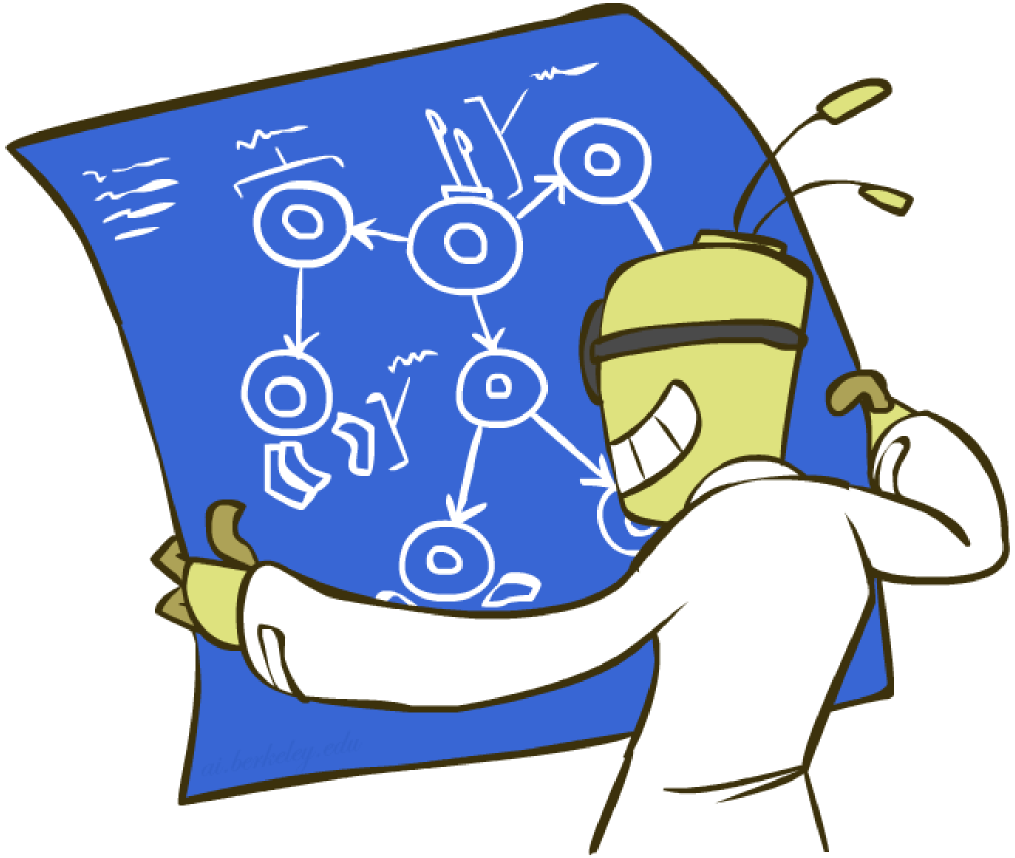
$T \perp\!\!\!\perp D | R, S$ No

Structure Implications

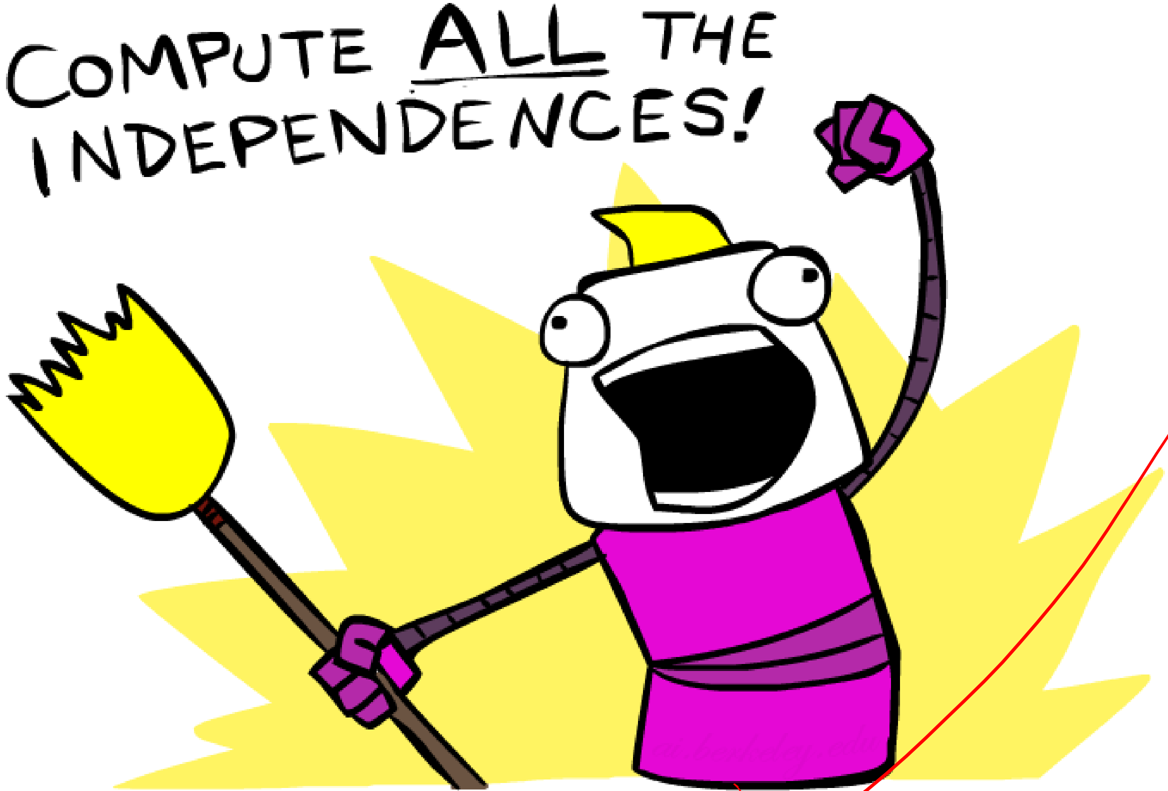
- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

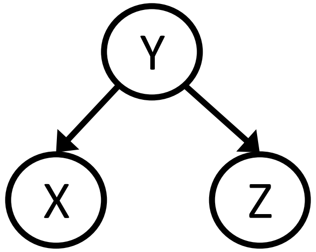
- This list determines the set of probability distributions that can be represented



Computing All Independences

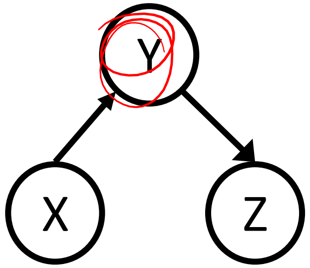


$P(x,y,z) = P(x)P(y|x)\,P(z|y)$

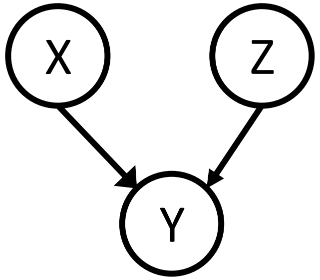


$X \perp\!\!\!\perp Z \mid Y$

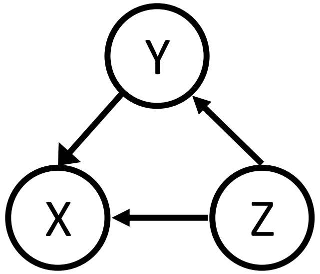
$P(z|x,y) = P(z|y)$



$X \perp\!\!\!\perp Z \mid Y$

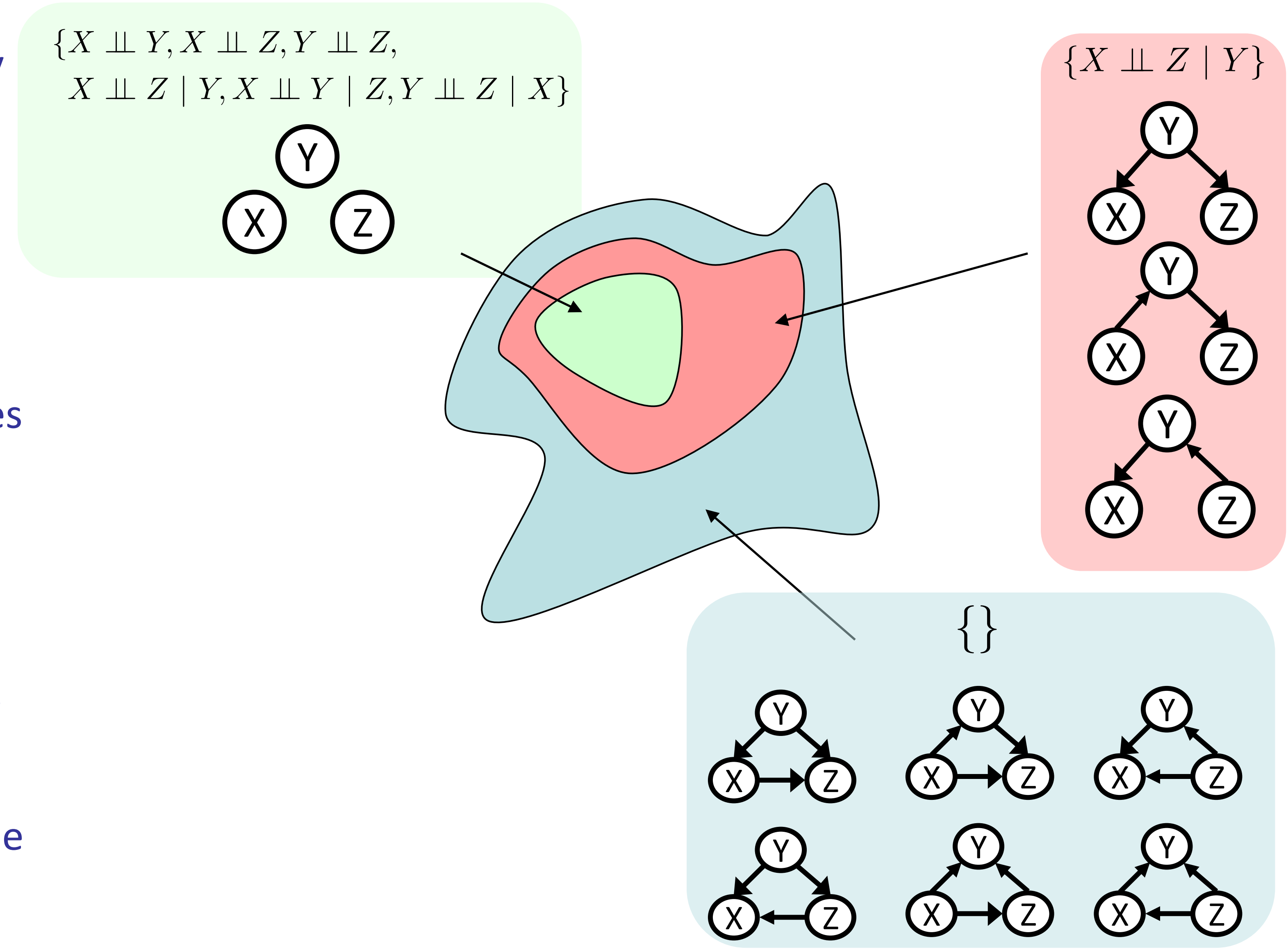


$X \perp\!\!\!\perp Z$



Topology Limits Distributions

- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution



Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

Bayes' Nets

- ✓ Representation
- ✓ Conditional Independences
- Probabilistic Inference
  - Enumeration (exact, exponential complexity)
  - Variable elimination (exact, worst-case exponential complexity, often better)
  - Inference is NP-complete
  - Sampling (approximate)
- Learning Bayes' Nets from Data