Alice decides that she wants to keep re-taking AI every semester for the rest of eternity (she *really* likes AI). We're interested in modeling whether she passes the class or not as a Markov chain. Suppose that in semester t she passes the class; then in semester $t+1$ she passes the class with probability 0.8 (maybe she gets bored and forgets to pay attention). On the other hand, if she doesn't pass in semester t then she'll pass with probability 0.4.

1. Suppose that in semester $t = 0$ Alice passes the class with probability 0.5. Compute the probability that she passes in semester $t = 1$ and semester $t = 2$.

The transition matrix is

$$T = \begin{bmatrix} p(P|P) & P(P|F) \\ p(F|P) & P(F|F) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$$

Her probability distribution at $t = 1$ is

$$t_1 = T * \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

Her probability distribution at $t = 2$ is

$$t_2 = T * t_1 = \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}$$

2. Compute the stationary distribution of this chain.

The stationary value is found from $t_\infty = T * t_\infty$.

$$\begin{aligned} \begin{bmatrix} t_{\infty 1} \\ t_{\infty 2} \end{bmatrix} &= T * \begin{bmatrix} t_{\infty 1} \\ t_{\infty 2} \end{bmatrix} \\ &= \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} * \begin{bmatrix} t_{\infty 1} \\ t_{\infty 2} \end{bmatrix} \end{aligned}$$

This leads to simulataneous equations.

$$\begin{aligned} t_{\infty 1} &= 0.8 * t_{\infty 1} + 0.4 t_{\infty 2} \\ t_{\infty 2} &= 0.2 * t_{\infty 1} + 0.6 t_{\infty 2} \end{aligned}$$

Both equations lead to $t_{\infty 1} = 2t_{\infty 2}$. Therefore the normalized probabilities are $[2/3 \; 1/3]'$.