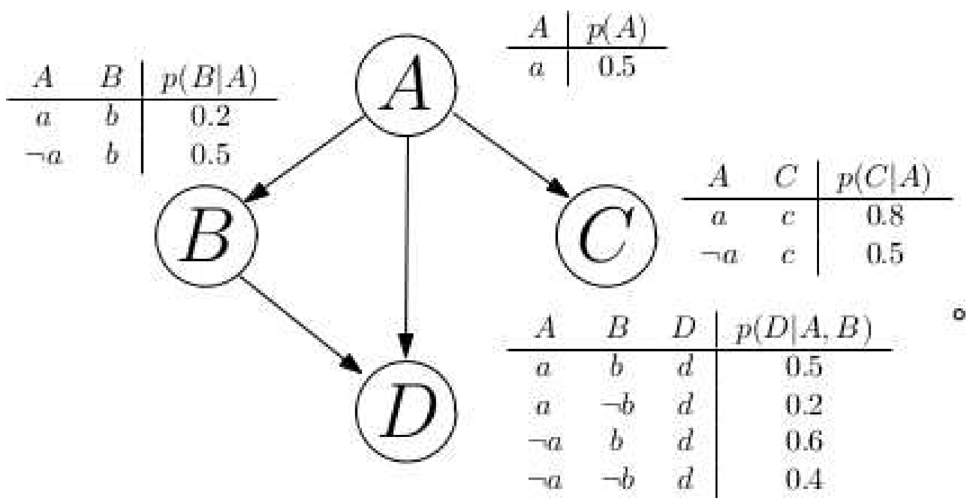
1 Bayes Nets (49pts)

Consider the graphical model below.



1.1 Variable Elimination (24pts)

Answer the query $p(A|\neg b)$ by variable elimination.

(4pts) A possible solution sequence eliminates the hidden variables in the order C, D. From the Bayes' net structure,

$$P(A|\neg b) = \alpha P(A) P(\neg b|A) \sum_d P(D|A, \neg b) \sum_c P(C|A)$$

- (4pts) There is only one factor with C, and summing it out yields $f_1(A) = 1$.

A	C	$P(C\|A)$
T	T	0.8
T	F	0.2
F	T	0.5
F	F	0.5

$\rightarrow$

A	$f_1(A)$
T	0.8+0.2=1
F	0.5+0.5=1

- (4pts) There is only one factor with D, and summing it out yields $f_2(A, \neg b) = 1$.

A	B	D	$P(D\|A, B)$
T	F	T	0.2
T	F	F	0.8
F	F	T	0.4
F	F	F	0.6

$\rightarrow$

A	B	$f_2(A, \neg b)$
T	F	1
F	F	1

- (8pts) Join on A to yield $f_3(A, \neg b) = P(A)P(\neg b|A)f_1(A)f_2(A) = P(A)P(\neg b|A)$.

A	$f_3(A, \neg b)$
T	$0.5 * 0.8 = 0.4$
F	$0.5 * 0.5 = 0.25$

- (4pts) Normalize to find $P(A|\neg b)$.

A	$P(B|\neg b)$
T	$0.4/0.65 = 8/13$
F	$0.25/0.65 = 5/13$

1.2 Sampling (25pts)

Answer the query $p(A|\neg b)$ by sampling as follows.

1. (15pts) Perform 5 rounds of sampling on this network (given the evidence that $B = \neg b$). Sample top-down and when you need to arbitrarily choose a variable to sample next, choose alphabetically. Use the following sequence of random values to perform this.

$$
\begin{array}{ccc}
0.15 & 0.95 & 0.48 \\
0.80 & 0.42 & 0.91 \\
0.79 & 0.85 & 0.93 \\
0.25 & 0.62 & 0.15 \\
0.49 & 0.93 & 0.05
\end{array}
\qquad
\begin{array}{rrrr}
a & \neg b & \neg c & \neg d \\
\neg a & \neg b & c & \neg d \\
\neg a & \neg b & \neg c & \neg d \\
a & \neg b & c & d \\
a & \neg b & c & d
\end{array}
$$

2. (5pts) For each sample, determine the weight using likelihood weighting.

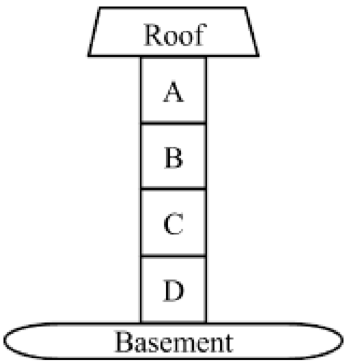
$$
\begin{array}{rrrrl}
a & \neg b & \neg c & \neg d & 0.8 \\
\neg a & \neg b & c & \neg d & 0.5 \\
\neg a & \neg b & \neg c & \neg d & 0.5 \\
a & \neg b & c & d & 0.8 \\
a & \neg b & c & d & 0.8
\end{array}
$$

3. (5pts) Now write down two estimates of the answer to the original query.

$$
\begin{aligned}
P(A|\neg b) &= \begin{bmatrix} P(a|\neg b) \\ P(\neg a|\neg b) \end{bmatrix} \\
&= \begin{bmatrix} \frac{0.8+0.8+0.8}{0.8+0.5+0.5+0.8+0.8} \\ \frac{0.5+0.5}{0.8+0.5+0.5+0.8+0.8} \end{bmatrix} \\
&= \frac{1}{3.4} \begin{bmatrix} 2.4 \\ 1.0 \end{bmatrix}
\end{aligned}
$$

2 HMMs (54pts)

You are part of the CS 6300 Search Team to find Waldo. Waldo randomly moves around floors A, B, C, and D. Waldo's location at time t is X_t. At the end of each timestep, Waldo stays on the same floor with probability 0.5, goes upstairs with probability 0.3, and goes downstairs with probability 0.2. If Waldo is on floor A, he goes down with probability 0.2 and stays put with probability 0.8. If Waldo is on floor D, he goes upstairs with probability 0.3 and stays put with probability 0.7. Waldo can only move one floor at a time; e.g., $P(D|A) = 0$. Waldo will not go onto the roof or into the basement.



X_0	$P(X_0)$
A	0.1
B	0.2
C	0.3
D	0.4

S_r	$P(S_r\|d_r)$	S_b	$P(S_b\|d_b)$
$+s$	$0.3 * d_r$	$+s$	$1 - 0.3 * d_b$
$-s$	$1 - 0.3 * d_r$	$-s$	$0.3 * d_b$

To aid the search a sensor S_r is installed on the roof and a sensor S_b is installed in the basement. Both sensors detect either sound ($+s$) or no sound ($-s$). The distribution of sensor measurements is determined by d, the number of floors between Waldo and the sensor. For example, if Waldo is on floor B, then $d_b = 2$ because there are two floors (C and D) between floor B and the basement and $d_r = 1$ because there is one floor (A) between floor B and the roof. Note that d_r or d_b might be zero if Waldo is adjacent to the respective sensor.

2.1 Forward Algorithm (36pts)

(6pts) At $t = 1$, Waldo potentially moves and the corresponding sensor readings are $S_r = +s, S_b = -s$. Estimate $P(X_1|S_r = +s, S_b = -s)$.

$$
\begin{aligned}
P(X_1|S_r = +s, S_b = -s) &= \alpha P(S_r = +s|X_1)P(S_b = -s|X_1)\sum_{x_0} P(X_1|x_0)P(x_0) \\
&= \alpha \begin{bmatrix} P(S_r = +s|A)P(S_b = -s|A) \\ P(S_r = +s|B)P(S_b = -s|B) \\ P(S_r = +s|C)P(S_b = -s|C) \\ P(S_r = +s|D)P(S_b = -s|D) \end{bmatrix} \\
&\quad \begin{bmatrix} P(A|A)P(A) + P(A|B)P(B) + P(A|C)P(C) + P(A|D)P(D) \\ P(B|A)P(A) + P(B|B)P(B) + P(B|C)P(C) + P(B|D)P(D) \\ P(C|A)P(A) + P(C|B)P(B) + P(C|C)P(C) + P(C|D)P(D) \\ P(d|A)P(A) + P(D|B)P(B) + P(D|C)P(C) + P(D|D)P(D) \end{bmatrix}
\end{aligned}
$$

(24pts) Substituting the sensor values, entries for A and D are zeroed out.

$$
\begin{aligned}
P(X_1|S_r = +s, S_b = -s) &= \alpha \begin{bmatrix} 0 * 0.9 \\ 0.3 * 0.6 \\ 0.6 * 0.3 \\ 0.9 * 0 \end{bmatrix} \\
&\quad \begin{bmatrix} P(A|A)P(A) + P(A|B)P(B) + P(A|C)P(C) + P(A|D)P(D) \\ P(B|A)P(A) + P(B|B)P(B) + P(B|C)P(C) + P(B|D)P(D) \\ P(C|A)P(A) + P(C|B)P(B) + P(C|C)P(C) + P(C|D)P(D) \\ P(d|A)P(A) + P(D|B)P(B) + P(D|C)P(C) + P(D|D)P(D) \end{bmatrix} \\
&= 0.18\alpha \begin{bmatrix} 0 \\ P(B|A)P(A) + P(B|B)P(B) + P(B|C)P(C) + P(B|D)P(D) \\ P(C|A)P(A) + P(C|B)P(B) + P(C|C)P(C) + P(C|D)P(D) \\ 0 \end{bmatrix}
\end{aligned}
$$

(6pts) Now substitute the transition values and initial state probabilities.

$$
\begin{aligned}
P(X_1|S_r = +s, S_b = -s) &= 0.18\alpha \begin{bmatrix} 0 \\ 0.2 * 0.1 + 0.5 * 0.2 + 0.3 * 0.3 + 0 * 0.4 \\ 0 * 0.1 + 0.2 * 0.2 + 0.5 * 0.3 + 0.3 * 0.4 \\ 0 \end{bmatrix} \\
&= 0.18\alpha \begin{bmatrix} 0 \\ 0.21 \\ 0.31 \\ 0 \end{bmatrix} \\
&= \frac{1}{0.52} \begin{bmatrix} 0 \\ 0.21 \\ 0.31 \\ 0 \end{bmatrix}
\end{aligned}
$$

2.2 Particle Filtering (18pts)

You decide to track Waldo by particle filtering with 3 particles. At time $t = 2$, the particles are at positions A, B and C.

1. (6pts) Without incorporating any sensory information, what is the probability that the particles will be resampled as B, B, and C respectively after time elapse?

$$P(B|A)P(B|B)P(C|C) = (0.2)(0.5)(0.5) = 0.05$$

2. (6pts) To decouple this from the previous question, assume the particles after time elapsing are at B, C and D, and the sensors observe $S_r = +s$ and $S_b = -s$. What are the particle weights given these observations?

Particle	Weight
$X_1 = B$	$P(S_r = +s\|d_r = 1)P(S_b = -s\|d_b = 2) = 0.18$
$X_2 = C$	$P(S_r = +s\|d_r = 2)P(S_b = -s\|d_b = 1) = 0.18$
$X_3 = D$	$P(S_r = +s\|d_r = 3)P(S_b = -s\|d_b = 0) = 0$

3. (6pts) To decouple this from the previous question, assume the particle weights in the following table. What is the probability the particles will be resampled as B, B and D?

Particle	Weight
$X = B$	0.1
$X = C$	0.6
$X = D$	0.3

$$0.1 * 0.1 * 0.3 = 0.003$$