Chapter 6

Boundary Integral Methods

In the previous sections we've utilized differential methods to solve our boundary value problems. In this section we explore integral methods. Let's consider our example problem,

$$\nabla^2 u = -f \tag{6.1}$$

with boundary conditions,

$$u = \bar{u} \quad \text{on } \Gamma_1 \tag{6.2}$$

and

$$\frac{\partial u}{\partial n} = \frac{\partial \bar{u}}{\partial n} \quad \text{on } \Gamma_2 \tag{6.3}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ describes the complete boundary.

Recall the method of weighted residuals, where we defined the residual R to measure the difference between our trial solution and the true solution. We studied various ways to select our trial solution such as to minimize R. In the weighted residual method, we required that,

$$\int_\Omega R w_i \, d\Omega = 0 \tag{6.4}$$

where w_i is a specially chosen function called the weighting function. The weighted residual formulation for our example problem is

$$\int_\Omega w_i [\nabla^2 \tilde{u} + f] \, d\Omega = 0 \tag{6.5}$$

where $\tilde{u}$ are our trial functions. For the Galerkin formulation of the finite element method, we chose the weighting functions to be the basis (or shape) functions themselves, such that the residual was formulated as,

$$R_i = \int_\Omega [\nabla N_i \nabla N_j \, d\Omega] u_j + \int_\Omega N_i f \, d\Omega \tag{6.6}$$

Since we require the residual to vanish over all elements, this gave us the familiar equation

$$R_i = a_{ij} u_j + f_i = 0 \tag{6.7}$$

where

$$(a_{ij}) = \int_{\Omega_E} \frac{b_i b_j + c_i c_j}{4A^2}\, d\Omega = \frac{b_i b_j + c_i c_j}{4A} \tag{6.8}$$

and for the right hand side we have, assuming constant sources,

$$f_i = \int_{\Omega_E} \frac{a_i + b_i x + c_i y}{2A} f\, d\Omega = \frac{Af}{3} \tag{6.9}$$

as before.

For the boundary integral method, we apply the method of weigthed residuals to our field problem plus to the boundary conditions. This leads to,

$$\int_\Omega w_i^1(\nabla^2 u - f)\, d\Omega + \int_\Omega w_i^2(u - \bar{u})\, d\Gamma + \int_\Omega w_i^3(\frac{\partial u}{\partial n} - \frac{\partial \bar{u}}{\partial n})\, d\Gamma = 0 \tag{6.10}$$

where w_i^1, w_i^2, w_i^3 are weighting functions (still to be determined). Previously, we showed how we could discretize the solution domain and choose basis functions that this would lead us to the Galerkin formulation for the finite element method. There are alternative choices for the weights, however, that lead us to a different formulation. Recall Green's second indentity,

$$\int_\Omega (u\nabla^2 v - v\nabla^2 u)\, d\Omega = \int_\Gamma (u\nabla v - v\nabla u)\, d\Gamma \tag{6.11}$$

let's apply it to the first term of our weighted residual equation,

$$\int_\Omega w_i^1(\nabla^2)\, d\Omega = \int_\Omega u\nabla^2 w_i^1\, d\Omega + \int_\Gamma w_i^1 \frac{\partial u}{\partial n}\, d\Gamma - \int_\Gamma u\frac{\partial w_i^1}{\partial n}\, d\Gamma \tag{6.12}$$

furthermore, we assign the other weights to be:

$$w_i^2 = \frac{\partial w_i^1}{\partial n} \tag{6.13}$$

and

$$w_i^3 = -w_i^1 \tag{6.14}$$

If we now substitute all these in we obtain,

$$\int_\Omega u\nabla^2 w_i^1\, d\Omega - \int_\Omega w_i^1 f\, d\Omega = \int_{\Gamma_2} w_i^1 \frac{\partial \bar{u}}{\partial n}\, d\Gamma - \int_{\Gamma_1} w_i^1 \frac{\partial u}{\partial n}\, d\Gamma + \int_{\Gamma_2} u\frac{\partial w_i^1}{\partial n}\, d\Gamma \int_{\Gamma_1} \bar{u}\frac{\partial w_i^1}{\partial n}\, d\Gamma \tag{6.15}$$

Now what we do is to specify w_i^1 such that the volume integral term can be integrated. If we can do this, then we can figure out a way to isolate u in terms of known quantities. The standard way of specifying w_i^1 is to require it to be the solution of,

$$\nabla^2 w_i^1 = -\delta_i \tag{6.16}$$

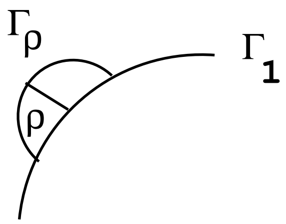

Figure 6.1: Discretization for the boundary element methods.

where δ_i is the Dirac delta function. Physically, this means that w_i^1 would represent the field corresponding to a point source at the origin and

$$\int_\Omega u \nabla^2 w_i^1 \, d\Omega = -u_i \tag{6.17}$$

where u_i represents the value of the unknown function at the singular point. Once we specify w_i^1 in this way and substitute it back into our equation, we can rewrite things such that u_i represents the value of our unknown function at an interior point.

$$\begin{gathered} u_i = \int_\Omega w_i^1 f \, d\Omega + \int_{\Gamma_2} w_i^1 \frac{\partial \bar{u}}{\partial n} \, d\Gamma - \\ \int_{\Gamma_1} w_i^1 \frac{\partial u}{\partial n} \, d\Gamma + \int_{\Gamma_2} u \frac{\partial w_i^1}{\partial n} \, d\Gamma \int_{\Gamma_1} \bar{u} \frac{\partial w_i^1}{\partial n} \, d\Gamma \end{gathered} \tag{6.18}$$

The solutions of the equation for w_i^1 are well know. For example, in a 3D isotropic media,

$$w = \frac{1}{4\pi r} \tag{6.19}$$

and for a 2D problem,

$$w = \frac{1}{2\pi} \ln \frac{1}{r} \tag{6.20}$$

As we can see from the last two equations, the field point becomes singular when $r = 0$ (i.e. when the field point is moved on to the boundary). In the simple case of a smooth boundary, we can distort the boundary slightly as shown in figure 6.1 and play tricks with the limits,

$$\int_{\Gamma_2} \frac{\partial w}{\partial n} u \, d\Gamma = \int_{\Gamma_2 - p} \frac{\partial w}{\partial n} u \, d\Gamma + \int_{\Gamma_p} \frac{\partial w}{\partial n} u \, d\Gamma \tag{6.21}$$

in the limit as $p \to 0$, we obtain,

$$\int_{\Gamma_p} \frac{\partial w}{\partial n} u \, d\Gamma = -\int_{\Gamma_p} \frac{u}{2\pi p} \, d\Gamma = -\frac{u}{2} \tag{6.22}$$

and

$$\int_{\Gamma_2} w \frac{\partial u}{\partial n} \, d\Gamma = \frac{1}{2\pi} \int_{\Gamma_2} \frac{\partial u}{\partial n} \ln \frac{1}{p} \, d\Gamma = 0 \tag{6.23}$$

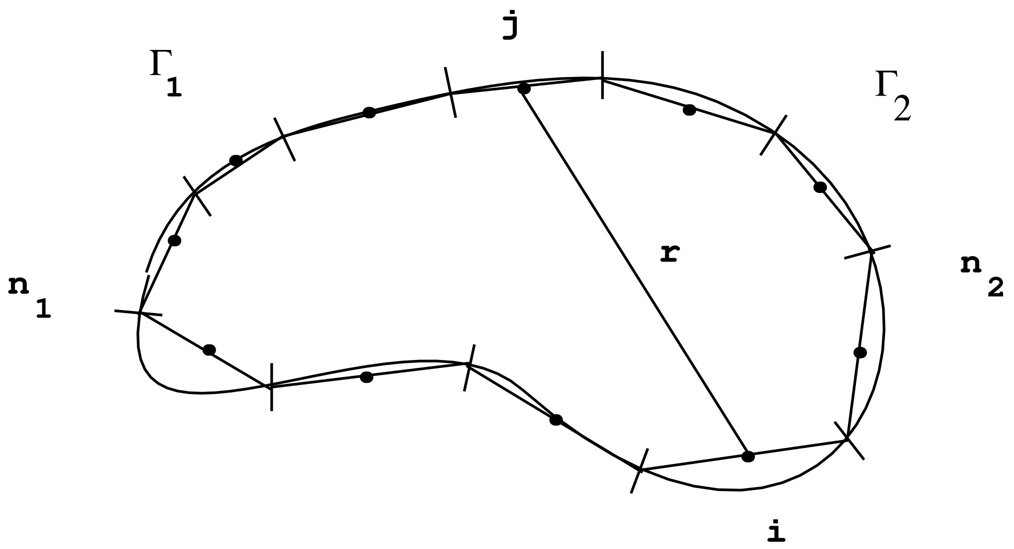

Figure 6.2: Discretization for the boundary element methods.

with similar results for a point on Γ_1. The limit adds an additional $u/2$ term to our equation. We can now, finally, write the general solution as:

$$cu_i - \int_\Omega f_i \, d\Omega + \int_{\Gamma_2} u \frac{\partial w_i^1}{\partial n} \, d\Gamma = \int_{\Gamma_1} w_i^1 \frac{\partial u}{\partial n} \, d\Gamma + \int_{\Gamma_2} w_i^1 \frac{\partial \bar{u}}{\partial n} \, d\Gamma - \int_{\Gamma_1} \bar{u} \frac{\partial w_i^1}{\partial n} \, d\Gamma \tag{6.24}$$

where $c = 1$ for an interior point and $c = 1/2$ for a point on a boundary.

Now we are at the point of discretizing the previous equation and forming a system of equations which we can solve. We discretize our geometry into N elements of which N_1 are along Γ_1 and N_2 are along Γ_2 as shown in figure 6.2. For a boundary element approximation which utilizes constant basis functions, we would have to assume that u and $\partial u / \partial n$ would be constant over each element. For linear functions, we would assume u and $\partial u / \partial n$ would vary linearly - and so one. Constant boundary elements have the advantage that the integrals in the previous equation can be evaluated analytically, which for any higher order scheme, we must use numerical integration techniques.

The discretization usually is written in the following rearranged way,

$$B_i + \sum_{j=1}^{N} H_{ij} u_j = \sum_{j=1}^{N} G_{ij} \frac{\partial u_j}{\partial n} \tag{6.25}$$

Here, B_i is the source term(s) that involves an integration over the volume for each node. For many problems, this is either a constant of zero (as in the case of solving Laplace's equation).

$$H_{ij} = \int_{\Gamma_j} \frac{\partial w}{\partial n} \, d\Gamma + \frac{1}{2} \delta_{ij} \tag{6.26}$$

and

$$G_{ij} = \int_{\Gamma_j} w \, d\Gamma \tag{6.27}$$

We note that N_1 values of u and N_2 values of $\partial u/\partial n$ are known, therefore there are precisely N unknowns in our equation to solve for. If we reorder the equations such that all the unknowns are on the left-hand side, we obtain a system of equations, $AX = Y$. In contrast to the finite element formulation, the boundary element coefficient matrix full and non-symmetric. An example of the matrix system, $AX = Y$ for $N = 5, N_1 = 2$ and $N_2 = 3$ is

$$\begin{pmatrix} H_{11} & H_{12} & -G_{13} & -G_{14} & -G_{15} \\ H_{21} & H_{22} & -G_{23} & -G_{24} & -G_{25} \\ H_{31} & H_{32} & -G_{33} & -G_{34} & -G_{35} \\ H_{41} & H_{42} & -G_{43} & -G_{44} & -G_{45} \\ H_{51} & H_{52} & -G_{53} & -G_{44} & -G_{55} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \frac{\partial u_3}{\partial n} \\ \frac{\partial u_4}{\partial n} \\ \frac{\partial u_5}{\partial n} \end{pmatrix} = \tag{6.28}$$

$$\begin{pmatrix} G_{11}\frac{\partial u_1}{\partial n} + G_{12}\frac{\partial u_2}{\partial n} - H_{13}u_3 - H_{14}u_4 - H_{15}u_5 - B_1 \\ G_{21}\frac{\partial u_1}{\partial n} + G_{22}\frac{\partial u_2}{\partial n} - H_{23}u_3 - H_{24}u_4 - H_{25}u_5 - B_2 \\ G_{31}\frac{\partial u_1}{\partial n} + G_{32}\frac{\partial u_2}{\partial n} - H_{33}u_3 - H_{34}u_4 - H_{35}u_5 - B_3 \\ G_{41}\frac{\partial u_1}{\partial n} + G_{42}\frac{\partial u_2}{\partial n} - H_{43}u_3 - H_{44}u_4 - H_{45}u_5 - B_4 \\ G_{51}\frac{\partial u_1}{\partial n} + G_{52}\frac{\partial u_2}{\partial n} - H_{53}u_3 - H_{54}u_4 - H_{55}u_5 - B_5 \end{pmatrix}. \tag{6.29}$$

Once the system of equations is solved for u and $\partial u/\partial n$ over the whole boundary, the interior points can be solved for by:

$$u_i = \sum_{i=j}^{N} \frac{\partial u_j}{\partial n} G_{ij} - \sum_{i=j}^{N} u_j H_{ij} \tag{6.30}$$

The integrals for H_{ij} and G_{ij} are usually solved for using Gauss quadrature. The numerical quadrature rules for evaluating element matrices have been extensively studied by several investigators (Carey and Oden, Stroud and Secrest, etc.).

We repeat the previous discuss from Chapter 4 here on numerical integration. Basically, we are interested in numerically evaluating an integral over a particular domain. For example, consider the one-dimensional integral

$$\int_{-1}^{1} f(t)dt = af(t_1) + bf(t_2) \tag{6.31}$$

such that we want to approximate the integral by right had side. To do so, we need to evaluate a, b, t_1, t_2. Let's assume that we want our approximation to be valid for integrands of degree three, i.e. it must satisfy $f(t) = t^3, f(t) = t^2, f(t) = 1$, and $f(t) = 1$. Substituing these into the integrals, we obtain

$$f(t) \quad = \quad t^3 \rightarrow \int_{-1}^{1} t^3 dt = 0 = at_1^3 + bt_2^3$$

$$
\begin{aligned}
f(t) &= t^2 \rightarrow \int_{-1}^{1} t^2 dt = \frac{2}{3} = a t_1^2 + b t_2^2 \\
f(t) &= t \rightarrow \int_{-1}^{1} t dt = 0 = a t_1 + b t_2 \\
f(t) &= 1 \rightarrow \int_{-1}^{1} dt = 2 = a + b
\end{aligned}
\tag{6.32}
$$

We can satisfy our original statement whith either $b = 0, t_2 = 0, t_1 = t_2$ or $t_1 = -t_2$. Only the last choice, $t_1 = -t_2$ yields numerically reasonable results. This give us $a = b = 1$ and $t_2 = t_1 = \sqrt{(1/3)}$. And our approximation becomes

$$\int_{-1}^{1} f(t)dt \approx f(-\sqrt{\frac{1}{3}}) + f(\sqrt{\frac{1}{3}}). \tag{6.33}$$

This simple approximation will yield very accurate values for integrands of degree three. In general, Gauss quadrature can be expressed as

$$\int_a^b f(x)w(x)dx \approx \sum_{i=0}^{n} A_i f(x_i) \tag{6.34}$$

where

$$A_i = \int_a^b w(x) \prod_{j=0}^{n} \frac{x - x_j}{x_i - x_j} dx \tag{6.35}$$

The $w(x)$ are the weights. Below are tables of quadrature points and weights for line segments and triangles.

1-D Gauss quadrature points and weights

Order	ξ_j	W_j
1	0	2
2	$\pm\sqrt{(1/3)}$	1
3	0	8/9
	$\pm\sqrt{(3/5)}$	5/9

Quadrature points and weights for triangles

Order	Points	Area Coords	Weights
Linear	1	$\xi_1 = \frac{1}{3}(1,1,1)$	1
Quadratic	3	$\xi_1 = \frac{1}{2}(1,1,0)$	
		$\xi_2 = \frac{1}{2}(0,1,1)$	1/3
		$\xi_3 = \frac{1}{2}(1,0,1)$	

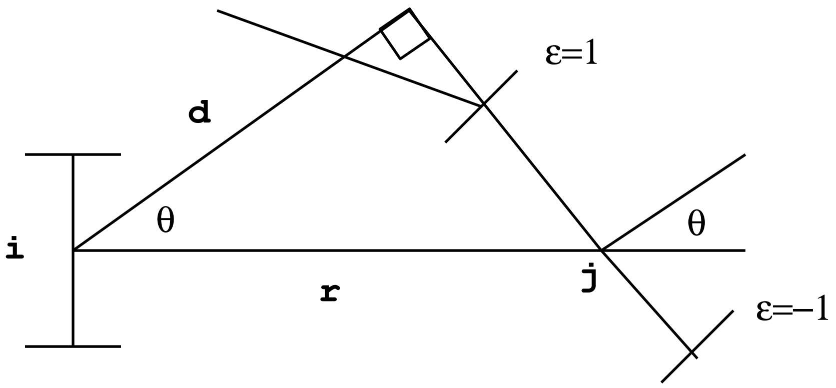

Figure 6.3: Geometry for element coefficients.

All the integrals can be calculated using 4 point Gauss quadrature, except the so-called self term ($i = j$). For this particular point, the integrand in the first term of H_{ij} is zero because r and n are orthogonal, i.e. $\frac{\partial(\ln 1/r)}{\partial n} = 0$, thus leaving only the singular term:

$$H_{ii} = \frac{1}{2} \tag{6.36}$$

and

$$G_{ii} = \int_{\Gamma_i} w d\Gamma = \frac{1}{2\pi} \int_{\Gamma_i} \ln \frac{1}{r} dr \tag{6.37}$$

As with finite elements, it is useful to change to a local coordinate system in which $r = L\xi$. Here, $2L$ is the lenght of the element and the collocation point is at the origin as shown in figure 6.3. How we have:

$$G_{ii} = \frac{1}{\pi} \int_0^b \ln \frac{1}{r} dr = \frac{L}{\pi} [\ln \frac{1}{L} + \int_0^1 \ln \frac{1}{\xi} d\xi] = \frac{L}{\pi} [\ln 1/L + 1] \tag{6.38}$$

For $i \neq j$, using Gauss quadrature, we have the following representations;

$$H_{ij} = \int \frac{\partial w}{\partial n} d\Gamma = \frac{L}{2\pi} \int \frac{\partial [\ln(\frac{1}{r})]}{\partial r} \frac{\partial r}{\partial n} d\xi = \tag{6.39}$$

$$-\frac{L}{2\pi} \int_{-1}^{1} \frac{d}{r^2} d\xi = -\frac{L}{\pi} \sum_{k=1}^{m} W_k \frac{d}{r^2(\xi_k)} \tag{6.40}$$

and

$$G_{ij} = \int w d\Gamma = \frac{L}{2\pi} \int_{-1}^{1} \ln(\frac{1}{r}) d\xi = \frac{L}{2\pi} \sum_{k=1}^{m} W_k \ln \frac{1}{\xi_k} \tag{6.41}$$

where W_k is the weight and ξ_k are the Gauss points in an m-point Gauss quadrature formula.

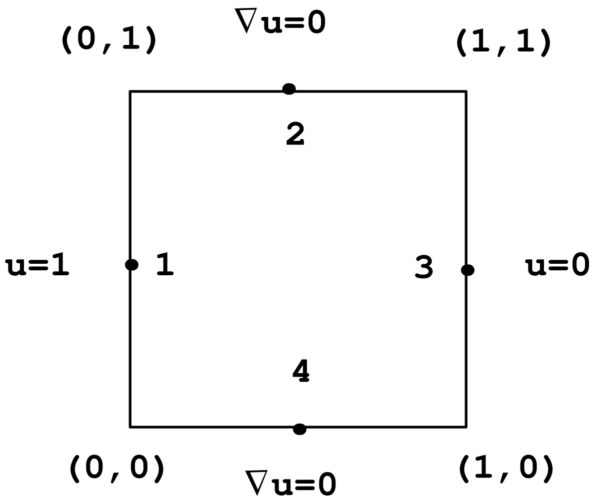

Figure 6.4: Geometry for the example boundary element problem.

6.0.2 2D Example Boundary Element Problem

Let's solve the steady-state heat equation on the unit square using the boundary element method. Thus we wish to solve

$$\nabla^2 u = 0 \quad \text{in } \Omega \tag{6.42}$$

with Dirichlet boundary conditions

$$u(0,y) = 1 \quad u(1,y) = 0 \tag{6.43}$$

and zero Neumann conditions

$$\frac{\partial u(x,0)}{\partial n} = \frac{\partial u(x,1)}{\partial n} = 0 \tag{6.44}$$

The boundary element formulation for this problem is:

$$\pi u_i = \sum_{j=1}^{4} u_j \int_{\Gamma_j} \frac{\partial \ln r_i}{\partial n} d\Gamma - \sum_{j=1}^{4} u_j \int_{\Gamma_j} \ln r_i d\Gamma \quad i = 1, 2 \ldots, N \tag{6.45}$$

We'll break up our domain into four elements (the sides of the square) as shown in figure 6.4.

Since we have four elements, we will obtain a 4×4 system of equations to solve from $\mathbf{H}u - \mathbf{G}\frac{\partial u}{\partial n} = 0$:

$$\begin{pmatrix} -G_{11} & H_{12} & -G_{13} & H_{14} \\ -G_{21} & H_{22} & -G_{23} & H_{24} \\ -G_{31} & H_{32} & -G_{33} & H_{34} \\ -G_{41} & H_{42} & -G_{43} & H_{44} \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial n} \\ u_2 \\ \frac{\partial u_3}{\partial n} \\ u_4 \end{pmatrix} = \begin{pmatrix} -H_{11} \\ -H_{21} \\ -H_{31} \\ -H_{41} \end{pmatrix}. \tag{6.46}$$

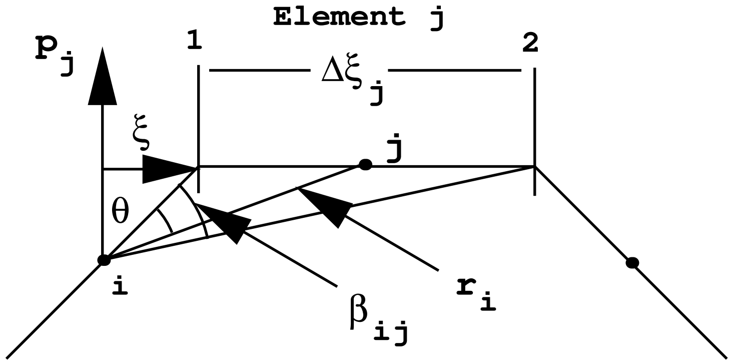

Figure 6.5: Geometry for direct integration.

The bulk of the work is calcuating the integrals for the coefficients H_{ij} and G_{ij}. We can use Gauss quadrature to compute the integrals, or we can calculate them directly. Refering to 6.5 we can calculate the coefficients as

$$G_{ij} = -\int \ln r_i d\Gamma = -\int_{\xi_1}^{\xi_2} \ln r_i d\xi = -\xi_2 \ln r_2 + \xi_1 \ln r_1 + \Delta\xi_j - p_j\beta_{ij}. \tag{6.47}$$

For example,

$$G_{11} = \Delta\xi_1 (\ln \frac{2}{\Delta\xi_1}) = (1)(0.6931 + 1) = 1.6931 \tag{6.48}$$

The H_{ij}s can be be computed in a similar way

$$H_{ij} = \int \frac{1}{r_i}\frac{\partial r_i}{\partial n} d\Gamma = \tan^{-1}\frac{\xi_2}{p_j} - \tan^1 \frac{\xi_1}{p_j} = \beta_{ij} \tag{6.49}$$

where β_{ij} can be obtained from geometric relationships,

$$\beta_{ij} = \cos^{-1}\frac{r_1^2 + f_2^2 - (\Delta\xi_j)^2}{2r_1r_2} \tag{6.50}$$

Thus one see that $H_{ii} = -\pi$ and $H_{12} = \beta_{12} = 1.1071$, etc. After computing all the integrals, we obtain,

$$\begin{pmatrix} 1.6931 & 1.1071 & 0.0727 & 1.1071 \\ 0.3349 & -\pi & 0.3349 & 0.9273 \\ 0.0727 & 1.1071 & 1.6931 & 1.1071 \\ 0.3349 & 0.9273 & 0.3349 & -\pi \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial n} \\ u_2 \\ \frac{\partial u_3}{\partial n} \\ u_4 \end{pmatrix} = \begin{pmatrix} \pi \\ -1.1071 \\ -0.9273 \\ -1.1071 \end{pmatrix}. \tag{6.51}$$

The boundary element solution is thus

$$\begin{pmatrix} \frac{\partial u_1}{\partial n} \\ u_2 \\ \frac{\partial u_3}{\partial n} \\ u_4 \end{pmatrix} = \begin{pmatrix} 1.2555 \\ 0.5000 \\ -1.2555 \\ 0.5000 \end{pmatrix}. \tag{6.52}$$

and the exact solution is

$$\begin{pmatrix} \frac{\partial u_1}{\partial n} \\ u_2 \\ \frac{\partial u_3}{\partial n} \\ u_4 \end{pmatrix} = \begin{pmatrix} 1.0000 \\ 0.5000 \\ -1.0000 \\ 0.5000 \end{pmatrix}. \tag{6.53}$$

thus the solution for u is more accurate than for $\frac{\partial u}{\partial n}$.