# Chapter 2

# Rate and Field Problems in Science and Engineering

As we mentioned previously, science and engineering systems are usually characterized as continuous or discrete. For the time being, we will concentrate our efforts on models of continuous systems. Within continuous systems, we can further characterize the systems as being described by either *rate* and/or *field* equations. Let's define what we mean by these terms.

## 2.1 Rate Equations

We assume that the system being modeled involves some continuous variable varying with respect to some other continuous variable and we have some reasonable hypothesis about the "rates of change" of the dependent variable with respect to the independent variable. When this change involves, say, one dependent variable (for example the temperature of coffee, $T$) depending on one independent variable (like $t$ for time), then this will yield a first order ordinary differential equation (o.d.e.) and is categorized as a **rate** equation. When the hypothesis involves the rate of change of the first derivative (e.g. $\frac{dT}{dt}$), then we obtain a second order o.d.e. (and so on). If there are a number of dependent continuous variables and only one indepedent variable, the model yields a system of first (or higher) order o.d.e.s [13]. Examples of rate equations are given below.

### 2.1.1 Population Growth Models

$$\frac{dP}{dt} = (b - d)P \tag{2.1}$$

where $P$ is the population at a given time, $t$, $b$ denotes the number of births, and $d$ the number of deaths. This has the solution,

$$P(t) = P_0 e^{(b-d)t} \tag{2.2}$$

When $(b-d) > 0$ there is exponential growth, when $(b-d) < 0$ there is decay and when $(b-d) = 0$ there is stagnation.

### 2.1.2 Newton's Law of Cooling

As we have already seen, Newton's Law of Cooling describes another first order rate equation,

$$\frac{dT}{dt} = -r(T - T_s) \tag{2.3}$$

with solution

$$T(t) = T_s + (T_0 - T_s)e^{-rt} \tag{2.4}$$

as before.

### 2.1.3 Diffusion - Fick's Law

The time rate of movement of a solute across a thin membrane is proportional to the area of the membrane and to the difference in concentrations of the solute on the two sides of the membrane. This diffusion process is described by another first order rate equation, which goes by the name of Fick's Law. If the area of the membrane is constant and the concentration on one side is kept fixed at $A$ while the concentration on the other side is initially $C_0 < A$, then Fick's law can be stated as

$$\frac{dC}{dt} = K(A - C) \tag{2.5}$$

### 2.1.4 Mechanics and Circuits

Similarly, damped harmonic oscillators and LCR circuits can be described by second order rate equations (or systems of second order rate equations as is often the case).

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{2.6}$$

for the displacement $x$, of a mechanical spring system of mass $m$, at the end of a spring of stiffness $k$ against a resistive drag force proportional to $b$. Likewise, an LCR circuit can be described by the exact same equation.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \tag{2.7}$$

Where $q$ is charge, $L$ is inductance, $R$ is resistance, and $C$ is the capacitance within the circuit.

## 2.2 Field Equations

If there is one dependent continuous variable (say the electric field vector, $E$, for example) and a number of independent continuous variables (usually these are space $x$,$y$,$z$ and time $t$, then the resulting model is described by a partial differential equation (p.d.e.) and characterized as a **field equation**. If there are a number of dependent continuous variables and a number of independent variables, the we obtain a system of p.d.e.s as our model [13]. Alternatively, a field can also be characterized as a scalar or vector function of space, a volume holding energy, or a finite or infinite set whose elements are assigned unique valuse of a certain quantity [14, 15]. For example, in electrodynamics both the electric, $\mathbf{E}$ and magnetic $\mathbf{B}$ field vectors are continuous functions of $(x, y, z, t)$ and furthermore, have continuous (we assume) first and second order partial derivatives of the continuous independent variables $(x, y, z, t)$. Other examples include the velocity, $v(x, y, z, t)$ and pressure $p(x, y, z, t)$ exhibited by a fluid, the temperature within a solid, $T(x, y, z, t)$ and so on.

Usually in the solution of engineering field problems, one first converts the laws governing the physical phenomena into partial differential equations. This is what is usually meant as **mathematical modeling**. One common way of performing this conversion is to consider an elementary volume element and apply to it continuity and conservation principles such as, momentum, heat, mass, energy, etc. While we will not be concentrating our efforts on mathematical modeling in this class, it is worthwhile to work through a few illustrative examples.

### 2.2.1 Mass Balance Equations

One of the most common ways of converting physical laws to partial differential equations is using a continuity principle, such as the continuity of mass or heat or charge. For example, consider a fluid passing through an element of volume, $d\tau$. Let the fluid have velocity $v$ and let $v_n$ be the normal component of the velocity of a fluid at any point on the surface $S$ of our volume element. The mass of fluid flowing out of the volume element in time $\Delta t$ across the surface, as shown in figure 2.1, is:

$$\Delta t \int\int_S \rho v_n dS = \Delta t \int\int_S \rho \mathbf{v} \cdot d\mathbf{S} \tag{2.8}$$

if we invoke Gauss' divergence theorem,

$$\int\int_S \mathbf{A} \cdot \mathbf{n} dS = \int\int\int_\tau \nabla \cdot \mathbf{A} d\tau \tag{2.9}$$

we obtain

$$\Delta t \int\int\int_\tau \nabla \cdot (\rho v) d\tau \tag{2.10}$$

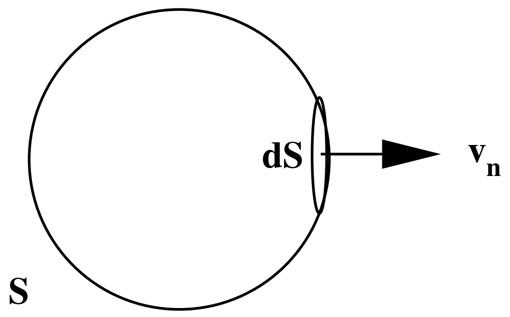


Figure 2.1: Mass of fluid flowing out of a volume element.

The change of mass of the fluid in the volume element $d\tau$ in time $\Delta t$ is:

$$-\Delta t\frac{\partial}{\partial t}\int\int\int_{\tau}\rho d\tau=-\Delta t\int\int\int_{\tau}\frac{\partial\rho}{\partial t}d\tau \tag{2.11}$$

Combining the last two equations, we obtain:

$$\int\int\int_{\tau}[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho v)]d\tau=0 \tag{2.12}$$

Since this must hold for all volumes, we have

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho v)=0 \tag{2.13}$$

Which is the continuity equation for fluid dynamics. We can obtain other representations for fluids by making some simplifications based on the physics of the situation. For example, if the fluid is incompressible, which means that $\rho$ = constant, then the divergence term is zero,

$$\nabla\cdot(\mathbf{v})=0 \tag{2.14}$$

Furthermore, if the fluid is irrotational, which means there exists a scalar velocity potential function $\Phi$ such that the vector velocity can be written as the gradient of the scalar function (such fields are called conservative),

$$\mathbf{v}=-\nabla\Phi \tag{2.15}$$

Then the equation for an incompressible, irrotational fluid can be obtain by combining the last two equations which reduces to Laplace's equation,

$$\nabla^2\Phi=0 \tag{2.16}$$

### 2.2.2 Continuity of Heat Flow

Similarly, we can conserve heat flow through an element of volume. While the names of the variables change, the procedure is the same. Let $v$ this time represent the heat flow velocity,

$\sigma$ be the thermal conductivity, $T$ denote temperature and $\rho$ density. Then we have

$$\int\int_S \mathbf{v} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int\int\int_\tau \sigma\rho T d\tau \tag{2.17}$$

From experiments, we have the empirical relationship

$$\mathbf{v} = -k\nabla T \tag{2.18}$$

where $k$ represents the diffusivity of the substance. If, for the moment, we assume that $\sigma$ and $\rho$ are constants, then we can obtain,

$$\sigma\rho \int\int\int_\tau \frac{\partial T}{\partial t} d\tau = \int\int_S k\nabla T \cdot d\mathbf{S} = \int\int\int_\tau \nabla \cdot (k\nabla T) d\tau \tag{2.19}$$

or, again, since this must hold for all volumes,

$$\sigma\rho \frac{\partial T}{\partial t} = \nabla \cdot (k\nabla T) \tag{2.20}$$

which is the heat equation. Furthermore, if $k$ is constant, this reduces to the more recognizable form:

$$\nabla^2 T = \frac{\sigma\rho}{k}\frac{\partial T}{\partial t} \tag{2.21}$$

If the temperature $T$ has reached steady state conditions, then we obtain the steady state heat equation,

$$\nabla^2 T = 0 \tag{2.22}$$

which is Laplace's equation again.

### 2.2.3 Conservation of Charge

One direct application of Gauss' law is its version in electrostatics. The charge within a closed spherical surface (a so-call *Gaussian surface*) is related to the electric field $\mathbf{E}$ normal to a surface element, $d\mathbf{S}$ by

$$\int\int_S \mathbf{E} \cdot d\mathbf{S} = 4\pi \int\int\int_\tau \rho d\tau \tag{2.23}$$

or

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{2.24}$$

which are the integral and differential forms of Gauss' law (one of the four Maxwell's equations for electrodynamics). In electrostatics, the electric field is conservative (this is also called solinoidal) which means that the curl of the electric field is zero,

$$\nabla \times \mathbf{E} = 0 \tag{2.25}$$

such that the vector field, $\mathbf{E}$ can be written as the gradient of a scalar potential function, $\Phi$,

$$\mathbf{E} = -\nabla \Phi \tag{2.26}$$

substituting this into the differential form of Gauss' law, we obtain,

$$\nabla \cdot (\nabla \Phi) = -4\pi\rho \tag{2.27}$$

or

$$\nabla^2 \Phi = -4\pi\rho \tag{2.28}$$

which is Poisson's equation. If there is no charge bounded within our Gaussian surface, then $\rho$ is zero and we, once again, obtain Laplace's equation:

$$\nabla^2 \Phi = 0 \tag{2.29}$$

## 2.3 Methods for Approximating Boundary Value Problems

In the next several chapters we will explore many of the methods for approximating science and engineering field problems. As we saw in the last chapter, many of the problems scientists and engineers tackle, many times have similar forms - if not identical, even though they relate to different physical phenomena. Take Poisson's equation, for example. It has variants through just about all of science and engineering:

| | |
|---|---|
| Electrostatics | $\nabla^2 \Phi = -\frac{\rho}{\epsilon}$ |
| Magnetics | $\nabla^2 A = -\mu J$ |
| Heat Flow | $\nabla^2 T = -\frac{q}{k}$ |
| Elastic Membranes | $T\nabla^2 u = F$ |
| Torsion | $\nabla^2 \Phi = -2G\theta$ |
| Fluid Flow | $\nabla^2 p = \frac{q}{\rho}$ |

When looking at the table of contents, one sees a list of several different methods. One might rightfully ask why we are going to study so many methods - isn't there just one "best" method? The answer to the "best" method question question is - yes, there *is* usually a "best" method for the problem, but it *depends* on parameters of the problem *which* of the methods is the "best" method. When scientists and engineers approach problems, they bring along their respective *toolboxes* or *toolsets*. These contain all the various techniques and "tricks of the trade" they have thus far assembled during their career. What has been observed is that, often times, there are a limited number of tools to choose from to carry out a specific job. For example, a particular engineer might have learned how to solve field problems using the finite difference method and only the finite difference method. Let's equate this to having a hammer (and only a hammer). Thus, when confronted with a field problem to solve, they will always use their hammer to solve it - even though another tool (like a screwdriver) might be more appropriate. One of my major goals in this course, is to provide you with a bigger toolbox and provide you with some guidelines on the appropriate use of those tools.

## 2.4 Classification of Second Order Field Problems

Consider the following (hopefully) familiar equations and their associated application area [16]:

1. Laplace's equation, $\nabla^2\Phi = 0$.
   - Heat flow
   - Electrostatics, magnetostatics, dielectrics, currents
   - Hydrodynamics
   - Gravitation
2. Poisson's equation, $\nabla^2\Phi = f$
   - See examples given in section 3.3.1.
3. The wave (Helmholtz) and time-independent diffusion equation, $\nabla^2\Phi \pm k^2\Phi = 0$.
   - Acoustics
   - Electromagnetic waves
   - Nuclear physics
   - Elastic waves in solids, vibrating membranes
4. The time-dependent diffusion equation, $\nabla^2\Phi = \frac{1}{a^2}\frac{\partial\Phi}{\partial t}$
5. The time-dependent wave equation, $\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}$
6. Maxwell's coupled partial differential equations for electric and magnetic fields.

All of these equations can be written in the form [17]

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \tag{2.30}$$

where $a, b, c, d, e, f$, and $g$ are given functions which are continuous in some region of space.

Two characteristics are particularly important:

1. All of these equations are linear in the unknown function $\Phi$. The pde is linear if $L$ can be expressed as a linear combination of $u$ and its derivatives.

2. All of the above equations are second order partial differential equations.

As observed, a great majority of science and engineering field problems can be formulated as linear (or quasi-linear), second-order partial differential equations. Thus, we will concentrate our efforts on approximating such field equations. [1]

General techniques for solving second-order field problems include [16]:

1. Separation of variables. The pde is split into ordinary differential equations which are then solved by the Frobenius (or are similar) method. This is the simplest method, but works only for simple geometries.

2. Integral solutions employing Green's functions.

3. Analytical and approximate methods using integral transform methods such as Fourier and Laplace.

4. Numerical techniques. This category includes methods such as

   - Finite difference methods
   - Finite element methods
   - Boundary integral/element methods
   - Multigrid and multilevel methods
   - Wavelet methods

   This is the category where we will concentrate our efforts.

The physical phenomenon described by a field equation is reflected in the behavior of its solution. The behavior of the solution, in turn, plays a major role in the selection of a numerical strategy. While it is perhaps somewhat tedious, going through the steps of formally classifying a system is a useful exercise. Consider a general quasi-linear second order partial differential equation [18],

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + e = 0 \tag{2.31}$$

[1] There are certainly many interesting and important problems in science and engineering which are described by higher-order partial differential equations and non-linear equations. However, due to our limited amount of time during the quarter, we will only be able to cover second-order linear equations.

where $u = u(x, y)$ and $a, b, c, f$ are functions of $x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ (I note that for time-dependent problems, t can be substituted for one of the spatial variables). To analyze this equation, we make the following change of variables:

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}, \quad r = \frac{\partial^2 u}{\partial x^2}, \quad s = \frac{\partial^2 u}{\partial x \partial y}, \quad t = \frac{\partial^2 u}{\partial y^2}. \tag{2.32}$$

Consider a parameterized curve $C$, $(x(\sigma), y(\sigma))$ in the $x - y$ plane. One can use the chain rule along $C$ to yield [18],

$$\frac{dp}{d\sigma} = \frac{\partial p}{\partial x}\frac{dx}{d\sigma} + \frac{\partial p}{\partial y}\frac{dy}{d\sigma} = r\frac{dx}{d\sigma} + s\frac{dy}{d\sigma}, \tag{2.33}$$

$$\frac{dq}{d\sigma} = \frac{\partial q}{\partial x}\frac{dx}{d\sigma} + \frac{\partial q}{\partial y}\frac{dy}{d\sigma} = s\frac{dx}{d\sigma} + t\frac{dy}{d\sigma}. \tag{2.34}$$

Combining these last two equations yields (after a bit of algebra),

$$s[a(\frac{dy}{dx})^2 - b(\frac{dy}{dx})^2 + c] - [a\frac{dp}{dx}\frac{dy}{dx} + c\frac{dq}{dx} + e\frac{dy}{dx}] = 0 \tag{2.35}$$

We now select the curve $C$ so that its slope $\frac{dy}{dx}$ at every point satisfies

$$a(\frac{dy}{dx})^2 - b(\frac{dy}{dx}) + c = 0. \tag{2.36}$$

Under this constraint, equation 2.35 reduces to a set of ordinary differential equations in either $x$ or $y$. Rather than worry about specific solutions to 2.35, we are more interested in the functional form of the solutions given by equation 2.36. Equation 2.36 is a quadratic equation in $\frac{dy}{dx}$. Consider the equation of a conic section,

$$ax^2 + bxy + c^2 + dx + ey + f = 0 \tag{2.37}$$

This equation describes an ellipse when $b^2 - 4ac < 0$, a parabola when $b^2 - 4ac = 0$, and a hyperbola when $b^2 - 4ac > 0$. Borrowing from this nomenclature, a field problem is categorized as either hyperbolic, parabolic, or elliptic according to whether the quantity $b^2 - 4ac$ is greater than, equal to, or less than, zero. Summarizing, second order field equations belong to one of the following classes:

| | |
|---|---|
| Elliptic | $b^2 - 4ac < 0$ |
| Parabolic | $b^2 - 4ac = 0$ |
| Hyperbolic | $b^2 - 4ac > 0$ |

While this is a somewhat artificial treatment, the interested reader can find more motivation by consulting [19, 20] for full details pertaining to how one arrives at the somewhat mystical value of $b^2 - 4ac$.

Physically, elliptic equations are usually associated with steady-state phenomena in which there are two or three independent variables. Examples of elliptic equations includes those of Laplace and Poisson,

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon}. \tag{2.38}$$

Parabolic equations arise from diffusion phenomena such as the flow of heat or fluids,

$$\nabla^2 T = -\frac{\sigma\rho}{k}\frac{\partial T}{\partial t}. \tag{2.39}$$

Hyperbolic equations are usually associated with wave phenomena or in the analysis of oscillation and vibration such as the wave equation and many equations in hydrodyanmics and acoustics,

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{2.40}$$