Formalizing the Real Numbers in

Homotopy Type Theory with Cubical Agda

by

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ABSTRACT

Real numbers in constructive mathematics have always seemed to require compromises of one form or another. Classical proofs of Cauchy completeness require countable choice, Bishop's setoid construction introduces persistent bookkeeping overhead on every definition and theorem, and Dedekind cuts force cumbersome universe-level tracking in predicative type theory. The Homotopy Type Theory (HoTT) book presents an alternative construction of the Cauchy real numbers as a higher inductive-inductive type family, avoiding all three compromises. We formalize the HoTT book reals in Cubical Agda, a proof assistant whose native support for higher inductive types allows the construction to be expressed directly. The code type-checks without postulates or holes, providing a foundation for further machine-assisted work in constructive analysis.

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1 - INTRODUCTION

Constructive mathematics endows mathematical proofs with a computational interpretation. When a constructive proof claims “given ε, there exists a δ,” it is expected to provide a method for obtaining such a δ. Given a specific ε, a reader should in principle be able to follow the proof step by step to determine a corresponding δ. By contrast, a classical proof is only required to show that a δ cannot fail to exist.

Constructive real analysis is attractive because it promises to narrow the gap between abstract existence claims and the explicit procedures needed to realize them in practice. Yet, giving a constructive account of real analysis becomes challenging almost immediately, because unlike elements of discrete number systems, arbitrary real numbers cannot in general be specified by finite data; in particular, equality of arbitrary real numbers is not effectively decidable. Prior constructions of the real numbers have always required compromises of one kind or another [1, p. 375].

To make the tension precise, consider the classical Cauchy construction of the real numbers. Classically, the Cauchy reals are obtained by taking the quotient of the Cauchy sequences of rational numbers under the equivalence relation that identifies sequences which are eventually arbitrarily close. The crucial step is to show Cauchy completeness: every Cauchy sequence of reals should have a limit in the reals. However, the classical proof of Cauchy completeness for the quotient fails in a constructive setting. We are given only a sequence of equivalence classes, but to build a representative for its limit we must choose a representative for each term of that sequence. Doing so requires the axiom of countable choice: for each term of the sequence we know the equivalence class is nonempty, but we must produce a function that selects a particular representative for each term. Constructively, the equivalence classes do not single out any particular representative, so no such function is available in general.

In his 1967 book *Foundations of Constructive Analysis*, Errett Bishop showed that a substantial portion of classical analysis could be recast constructively [2]. His aim was "to give a numerical meaning to as much as possible of classical abstract analysis" [2, p. 3]. Bishop's solution to the problem of Cauchy completeness is to avoid taking the quotient and to work instead with the setoid of Cauchy sequences of rational numbers, where a setoid is a set equipped with a canonical equivalence relation. This avoids the need for countable choice, but it introduces a persistent bookkeeping burden: because compatibility with the equivalence relation is not automatic, the same kinds of congruence proofs must be supplied repeatedly, often for routine constructions. Taken to its limit, this approach would force all of abstract algebra to be reformulated in terms of setoids in order to apply results to the reals, as observed by Gilbert [3].

Another alternative is to work with Dedekind reals, but these can be awkward in predicative type theory because their construction involves Dedekind cuts—subsets valued in a universe of propositions. The real numbers defined in this way thus live one universe level above their cuts, and a property of the reals one level higher still. Tracking these varying universe levels throughout a development becomes cumbersome [1, p. 377].

In the final chapter of the *Homotopy Type Theory* (HoTT) book, the authors present a new formulation of the Cauchy reals as a higher inductive-inductive type family [1]. We refer to this construction as *the HoTT book reals*, to distinguish it from the naive quotient of Cauchy sequences discussed previously. This construction is compelling because it avoids the compromises of the previous approaches. First, it builds Cauchy completeness directly into the definition by including a limit constructor for Cauchy approximations valued in the reals. The accompanying closeness relation is defined simultaneously with the type of reals itself, so this construction is not circular. This avoids the need for countable choice. Second, unlike in Bishop's work, it aligns the identity type of the reals with their intended notion of equality: the HoTT book reals come equipped with a path constructor identifying any two reals that are ε-close for every rational $\varepsilon > 0$. Finally, because this construction is not defined in terms of proposition-valued Dedekind cuts, it avoids the associated burden of tracking universe levels in predicative type theory.

Gilbert formalized the HoTT book reals in a modified version of the Rocq proof assistant and even generalized the construction to premetric spaces, following earlier work of O'Connor [3], [4]. However, Rocq by default lacks support for both inductive-inductive type families and higher inductive types. Gilbert worked with an experimental branch of Rocq that added inductive-inductive type families, but higher inductive types were still unavailable, so the path constructors for the reals and the accompanying closeness relation had to be postulated. As a result, the formalization could not be type checked in mainline Rocq, and the corresponding path constructors did not enjoy native computation rules.

In this thesis, we present a formalization of the HoTT book reals in Cubical Agda, whose native support for higher inductive types allows the construction to be expressed directly, without postulates. Cubical Agda implements cubical type theory, which gives a constructive interpretation of both higher inductive types and univalence, making it a natural setting for this work [5]. Our formalization follows Section 11.3 of the HoTT book. The code is open source and available at `https://github.com/utahplt/hott-reals`.

Chapter 2 presents the structure of the formalization in four stages. It begins with the higher inductive-inductive definition of the HoTT book reals and several variations of induction and recursion principles used throughout the development. It then develops a collection of results concerning the closeness relation and an alternative characterization thereof. From there, the chapter establishes results on lifting Lipschitz and nonexpanding maps to the reals and finally develops the algebraic and order structure of the HoTT book reals, culminating in the proof that reals form an Archimedean ordered field.

Chapter 3 reflects on what the Agda code revealed about the mathematical construction in Chapter 2. In several places, the process of formalization sharpened our understanding of the informal presentation. We articulated the hypotheses needed to extend multi-variable identities from the rationals to the reals, repaired a naive transcription of the enhanced recursion principle to a form stronger than an informal reading suggests, and recognized the structural necessity of the alternative characterization of closeness. The chapter also includes a demonstration that real arithmetic computes definitionally on rational inputs, an overview of the codebase organization, and a discussion of our experimentation with Claude Code during development.

Chapter 4 presents related work. Chapter 5 outlines future work. Chapter 6 concludes.

The Cubical Agda code typechecks without postulates or holes using the latest standard library release, and is available for other researchers to build on. The development clarifies how the HoTT book reals behave in a proof assistant with native support for higher inductive types. It provides a foundation for further machine-assisted work in constructive analysis.

2 - REAL NUMBERS AS A HIGHER INDUCTIVE TYPE

This chapter presents the mathematical development of the HoTT book reals ($\mathbb{R}$), beginning with their higher inductive-inductive definition and ultimately concluding that they form an Archimedean ordered field. We follow the notation and conventions of the HoTT book [1] and assume familiarity with basic Homotopy Type Theory. Readers less familiar with HoTT should still be able to follow the main ideas by consulting the HoTT book occasionally when unfamiliar concepts or notation arise.

The chapter is organized as follows. In Section 2.1, we introduce the definition of the HoTT book reals and specify their induction and recursion principles. We then develop the basic theory of the closeness relation in Section 2.2, and use it to state extension principles for Lipschitz and non-expanding maps in Section 2.3. Finally, in Section 2.4, we assemble the resulting algebraic and order-theoretic structure. The first three sections follow the HoTT book closely. In the final section, we continue to use the HoTT book for the basic algebraic constructions, but follow Gilbert [3] for the treatment of strict order and for the construction of multiplication and reciprocal, with Kraus's theorem on maps out of propositional truncations into sets providing a convenient formulation of one local-to-global step in the latter construction [6].

By default, we do not reproduce proofs already given in the HoTT book or Gilbert. We repeat an existing proof only when its content contributes meaningfully to the conceptual narrative, or when our proof strategy differs from the one given in those sources.

2.1 - DEFINITION, INDUCTION, AND RECURSION

The HoTT book reals are an adaptation of the classical Cauchy construction, in which the real numbers are obtained by adjoining limits to the rationals. The completion of a general metric space is obtained by adjoining limits of sequences satisfying the Cauchy condition, that is, sequences whose terms eventually become arbitrarily close. However, metric spaces presuppose a notion of distance valued in the real numbers, so proceeding in this way would be circular here, since we are in the process of constructing the reals themselves. The key observation is that the full structure of a metric is not required to formulate the Cauchy condition; it suffices to express approximate closeness between reals. In the HoTT book construction, this is captured by a family of binary closeness relations on the reals, indexed by positive rationals.

The main difficulty is that the definitions of the reals and the closeness relation depend on each other. The limit constructor for the reals takes as input a family of real approximations satisfying the Cauchy condition formulated in terms of closeness. Conversely, the closeness relation is defined by cases on the point constructors for the reals: rational-rational, rational-limit, limit-rational, and limit-limit. This kind of mutual dependence is handled by the framework of inductive-inductive definitions, which allow the simultaneous definition of a type A together with a type family $B : A \to \mathcal{U}$ over A [7]. In the present case, this simultaneously defined family takes the form of a ternary relation: for reals $u, v : \mathbb{R}$ and a positive rational ε, the type $u \sim_\varepsilon v$ expresses that u and v are ε-close.

In addition to adjoining limits, we must also ensure that arbitrarily close reals are identified. Specifically, reals which are ε-close for every positive rational ε ought to be treated as indistinguishable. This requirement is built into the definition via a path constructor that provides a path between any two reals satisfying this condition. For this reason, the HoTT book reals and the associated closeness relation are defined as a higher inductive-inductive type family.

To express the Cauchy condition in terms of the closeness relation, the HoTT book uses Cauchy approximations rather than traditional Cauchy sequences. A **Cauchy approximation** is a map $x : \mathbb{Q}_+ \to \mathbb{R}$ satisfying $x_\varepsilon \sim_{\varepsilon+\delta} x_\delta$ for all positive rationals ε and δ. Once closeness is indexed by positive rationals, it is natural for families of approximations to share the same index. Intuitively, a Cauchy approximation assigns, for each requested precision $\varepsilon > 0$, an approximate value of the intended limit. In contrast, an ordinary Cauchy sequence $\mathbb{N} \to \mathbb{R}$ does not by itself indicate which term to use for a requested precision; a modulus of convergence is also needed. Cauchy approximations therefore package a Cauchy sequence together with its modulus of convergence into a form suited to the closeness-based formulation of completion.

With this motivation in place, we can state the higher inductive-inductive definition of the HoTT book reals and their closeness relation.

Definition 2.1.1 (The Univalent Foundations Program [1, Definition 11.3.2])**.** The type $\mathbb{R}$ of **HoTT book reals**, together with a type family $\sim : \mathbb{Q}_+ \to \mathbb{R} \to \mathbb{R} \to \mathcal{U}$ referred to as the **closeness relation**, are defined simultaneously as the following higher inductive-inductive type family. The type $\mathbb{R}$ comes equipped with the following constructors:

- For each rational $q : \mathbb{Q}$, there is an element $\mathsf{rational}(q) : \mathbb{R}$.

- For each map $x : \mathbb{Q}_+ \to \mathbb{R}$ equipped with an element of type

$$\mathsf{IsCauchy}(x) :\equiv \prod_{(\varepsilon,\delta:\mathbb{Q}_+)} x_\varepsilon \sim_{\varepsilon+\delta} x_\delta,$$

  there is an element $\mathsf{limit}(x) : \mathbb{R}$. We call x a **Cauchy approximation**.

- For each $u, v : \mathbb{R}$ such that $u \sim_\varepsilon v$ for all $\varepsilon : \mathbb{Q}_+$, there is a path $\mathsf{path}(u, v) : u = v$.

The closeness relation $\sim : \mathbb{Q}_+ \to \mathbb{R} \to \mathbb{R} \to \mathcal{U}$ comes equipped with the following constructors:

- For all rationals q, r, ε with $\varepsilon > 0$, if $-\varepsilon < q - r < \varepsilon$ then $\mathsf{rational}(q) \sim_\varepsilon \mathsf{rational}(r)$.

- For all rationals q, ε, δ with $\varepsilon > 0$, $\delta > 0$, and $\varepsilon - \delta > 0$ and all Cauchy approximations $y : \mathbb{Q}_+ \to \mathbb{R}$, if $\mathsf{rational}(q) \sim_{\varepsilon-\delta} y_\delta$ then $\mathsf{rational}(q) \sim_\varepsilon \mathsf{limit}(y)$.

- For all Cauchy approximations $x : \mathbb{Q}_+ \to \mathbb{R}$ and all rationals r, ε, δ with $\varepsilon > 0$, $\delta > 0$, and $\varepsilon - \delta > 0$, if $x_\delta \sim_{\varepsilon-\delta} \mathsf{rational}(r)$ then $\mathsf{limit}(x) \sim_\varepsilon \mathsf{rational}(r)$.

- For all Cauchy approximations x, y and all rationals $\varepsilon, \delta, \eta$ with $\varepsilon, \delta, \eta > 0$ and $\varepsilon - (\delta + \eta) > 0$, if $x_\delta \sim_{\varepsilon-(\delta+\eta)} y_\eta$ then $\mathsf{limit}(x) \sim_\varepsilon \mathsf{limit}(y)$.

- Given reals u, v and a rational $\varepsilon > 0$, if $\varphi, \varphi' : u \sim_\varepsilon v$ then there is a path $\varphi = \varphi'$.

The $\mathsf{rational}$ constructor of $\mathbb{R}$ embeds each rational number as a real number. The limit constructor asserts that every Cauchy approximation $x : \mathbb{Q}_+ \to \mathbb{R}$ determines a real number $\mathsf{limit}(x)$, which we refer to as the limit of x. The remaining constructor, discussed below, identifies reals that are arbitrarily close.

Heuristically, x_ε is intended to be an approximation of $\mathsf{limit}(x)$ corresponding to the error tolerance ε. Assuming the limit exists, informal reasoning with the triangle inequality suggests that x_ε and x_δ are $(\varepsilon + \delta)$-close. The $\mathsf{IsCauchy}$ condition can be understood as internalizing this reasoning while avoiding any circular reference to the limit itself. A formal definition of the triangle inequality in the context of the closeness relation is given in Definition 2.2.1.

In the Agda formalization, a proof $\varphi : \mathsf{IsCauchy}(x)$ is tracked explicitly, but in our informal presentation here, we follow [1] and abuse notation slightly by writing $\mathsf{limit}(x)$ instead of $\mathsf{limit}(x, \varphi)$.

The path constructor enforces that any two reals that are ε-close for every positive rational ε are equal. In the HoTT book, this property is called **separatedness** [1, §11.3.2]. The terminology is analogous to the Hausdorff condition in topology, which classically says that distinct points can be separated by disjoint neighborhoods. For the real numbers, this matches the familiar metric intuition that distinct points are separated by a positive distance. Constructively, we use the corresponding positive formulation: if two reals are arbitrarily close, then they are equal.

The first constructor for closeness lifts the usual condition for ε-closeness for rational numbers to the reals. The next three constructors describe how closeness interacts with limits and can be understood using the heuristic explanation above: we think of x_ε as an approximation to $\mathsf{limit}(x)$ associated with the error tolerance ε, and then reason informally via the triangle inequality. The final constructor asserts that closeness is a proposition, that is, given $u, v : \mathbb{R}$ and $\varepsilon > 0$, any two proofs of $u \sim_\varepsilon v$ are equal.

The use of a higher inductive-inductive type family, together with the large number of constructors in Definition 2.1.1, makes the induction principle quite involved. Because $\mathbb{R}$ and $\sim$ are defined mutually, it is not sufficient in general to work with a type family only over $\mathbb{R}$ or only over $\sim$, though we will consider these as special cases shortly.

We will work up to the full induction principle following Forsberg [7, §3.2.5]. Forsberg makes a distinction between "simple" and "general" elimination principles for inductive-inductively defined types. Suppose $A : \mathcal{U}$ and $B : A \to \mathcal{U}$ are inductive-inductively defined types. Simple elimination rules use motives of the form

$$P : A \to \mathcal{U},$$
$$Q : \prod_{(x:A)} B(x) \to \mathcal{U}$$

and therefore have the form

$$\mathsf{induction}'_A : \cdots \to \prod_{(x:A)} P(x),$$
$$\mathsf{induction}'_B : \cdots \to \prod_{(x:A)} \prod_{(y:B(x))} Q(x, y).$$

In contrast, general elimination principles use motives of the form

$$P : A \to \mathcal{U},$$
$$Q : \prod_{(x:A)} \prod_{(y:B(x))} P(x) \to \mathcal{U}$$

and therefore have the form

$$\mathsf{induction}_A : \cdots \to \prod_{(x:A)} P(x),$$
$$\mathsf{induction}_B : \cdots \to \prod_{(x:A)} \prod_{(y:B(x))} Q(x, y, \mathsf{induction}_A(\ldots, x)).$$

With a general induction principle, the motive Q may depend on the result of induction for P. When constructing data over an indexed family, we often need access not only to the index, but also to the data already assigned to that index by induction.

The present case is slightly more elaborate, since the closeness relation is indexed by two reals and a positive rational [7, §6.2]. The codomain families in our setting therefore take the form

$$A : \mathbb{R} \to \mathcal{U},$$
$$B : \prod_{(u,v:\mathbb{R})} A(u) \to A(v) \to \prod_{(\varepsilon:\mathbb{Q}_+)} u \sim_\varepsilon v \to \mathcal{U}.$$

The family B may be understood as an ε-indexed binary relation between the fibers $A(u)$ and $A(v)$, defined only when $u, v : \mathbb{R}$ are known to be ε-close. Accordingly, we usually write this type using infix notation as $(u, a) \frown_\varepsilon^\varphi (v, b)$, where $a : A(u)$, $b : A(v)$, and $\varphi : u \sim_\varepsilon v$. Because closeness is a mere relation, the witness φ is often irrelevant, and the endpoints u and v are usually clear from context, so we will often abbreviate this as $a \frown_\varepsilon b$.

Definition 2.1.2 (The Univalent Foundations Program [1, § 11.3.2])**.** Assume codomain families A and $\frown$ as above. The general induction principle for the HoTT book reals, **$(\mathbb{R}, \sim)$-induction**, asserts that to construct sections of the families A and B, it suffices to specify data corresponding to each constructor of the higher inductive-inductive definition. Concretely, the required hypotheses are as follows:

- For each $q : \mathbb{Q}$, an element $f_{\mathsf{rational}}(q) : A(\mathsf{rational}(q))$.
- For each Cauchy approximation $x : \mathbb{Q}_+ \to \mathbb{R}$ and each $a : \prod_{(\varepsilon:\mathbb{Q}_+)} A(x_\varepsilon)$ satisfying

$$\prod_{(\varepsilon,\delta:\mathbb{Q}_+)} (x_\varepsilon, a_\varepsilon) \frown_{\varepsilon+\delta} (x_\delta, a_\delta), \tag{2.1}$$

  an element $f_{\mathsf{limit}}(x, a) : A(\mathsf{limit}(x))$. Whenever such an a satisfies (2.1), we refer to it as a **dependent Cauchy approximation** over x. As with $\mathsf{limit}(x)$, we typically suppress the witness that x is a Cauchy approximation and that a is a dependent Cauchy approximation when they are clear from context. Otherwise, we explicitly write $f_{\mathsf{limit}}(x, \varphi, a, \psi)$, for witnesses φ, ψ.

- For all $u, v : \mathbb{R}$ such that $u \sim_\varepsilon v$ for every positive rational ε, and all $a : A(u)$ and $b : A(v)$ such that $(u, a) \frown_\varepsilon (v, b)$ for every positive rational ε, a dependent path[1] $a =^A_{\mathsf{path}(u,v)} b$.
- For all $q, r : \mathbb{Q}$ and all $\varepsilon : \mathbb{Q}_+$, a proof that if $-\varepsilon < q - r < \varepsilon$ then

$$(\mathsf{rational}(q), f_{\mathsf{rational}}(q)) \frown_\varepsilon (\mathsf{rational}(r), f_{\mathsf{rational}}(r)).$$

- For all rationals $q : \mathbb{Q}$, all $\varepsilon, \delta : \mathbb{Q}_+$ with $\varepsilon - \delta > 0$, all Cauchy approximations $y : \mathbb{Q}_+ \to \mathbb{R}$, and all dependent Cauchy approximations b over y, a proof that if

[1]As in [1, §6.2], if $u : B(x)$, $v : B(y)$, and $p : x = y$, the type of dependent paths from u to v over p can be expressed $u =^B_p v := \mathsf{transport}^B(p, u) = v$

$\mathsf{rational}(q) \sim_{\varepsilon-\delta} y_\delta$ and $(\mathsf{rational}(q), f_{\mathsf{rational}}(q)) \frown_{\varepsilon-\delta} (y_\delta, b_\delta)$ then

$$(\mathsf{rational}(q), f_{\mathsf{rational}}(q)) \frown_\varepsilon (\mathsf{limit}(y), f_{\mathsf{limit}}(y, b)).$$

- For all Cauchy approximations $x : \mathbb{Q}_+ \to \mathbb{R}$, all dependent Cauchy approximations a over x, all $r : \mathbb{Q}$, and all ε, δ with $\varepsilon - \delta > 0$, a proof that if $x_\delta \sim_{\varepsilon-\delta} \mathsf{rational}(r)$ and $(x_\delta, a_\delta) \frown_{\varepsilon-\delta} (\mathsf{rational}(r), f_{\mathsf{rational}}(r))$ then

$$(\mathsf{limit}(x), f_{\mathsf{limit}}(x, a)) \frown_\varepsilon (\mathsf{rational}(r), f_{\mathsf{rational}}(r)).$$

- For all Cauchy approximations $x, y : \mathbb{Q}_+ \to \mathbb{R}$, all dependent Cauchy approximations a and b over x and y, respectively, and all $\varepsilon, \delta, \eta : \mathbb{Q}_+$ with $\varepsilon - (\delta + \eta) > 0$, a proof that if $x_\delta \sim_{\varepsilon-(\delta+\eta)} y_\eta$ and $(x_\delta, a_\delta) \frown_{\varepsilon-(\delta+\eta)} (y_\eta, b_\eta)$ then

$$(\mathsf{limit}(x), f_{\mathsf{limit}}(x, a)) \frown_\varepsilon (\mathsf{limit}(y), f_{\mathsf{limit}}(y, b)).$$

- For all $u, v : \mathbb{R}$, all $\varepsilon : \mathbb{Q}_+$, all $a : A(u)$ and $b : A(v)$, and all $\varphi : u \sim_\varepsilon v$, a proof that the type $(u, a) \frown_\varepsilon^\varphi (v, b)$ is a proposition.

Under these hypotheses, we obtain functions

$$\mathsf{induction}_{\mathbb{R}} : \prod_{(u:\mathbb{R})} A(u),$$

$$\mathsf{induction}_{\sim} : \prod_{(u,v:\mathbb{R})} \prod_{(\varepsilon:\mathbb{Q}_+)} \prod_{(\varphi:u\sim_\varepsilon v)} (u, \mathsf{induction}_{\mathbb{R}}(u)) \frown_\varepsilon^\varphi (v, \mathsf{induction}_{\mathbb{R}}(v))$$

which satisfy the following computation rules

$$\mathsf{induction}_{\mathbb{R}}(\mathsf{rational}(q)) \doteq f_{\mathsf{rational}}(q),$$

$$\mathsf{induction}_{\mathbb{R}}(\mathsf{limit}(x,\varphi)) \doteq f_{\mathsf{limit}}(x,\varphi,\varepsilon \mapsto \mathsf{induction}_{\mathbb{R}}(x_\varepsilon),\psi),$$

where $q : \mathbb{Q}$, $x : \mathbb{Q}_+ \to \mathbb{R}$, φ witnesses that x is a Cauchy approximation, and for $\varepsilon, \delta : \mathbb{Q}_+$, we define $\psi(\varepsilon,\delta) := \mathsf{induction}_{\sim}(x_\varepsilon, x_\delta, \varepsilon+\delta, \varphi(\varepsilon,\delta))$. Thus ψ witnesses that the induced function $\varepsilon \mapsto \mathsf{induction}_{\mathbb{R}}(x_\varepsilon)$ is a dependent Cauchy approximation over x.

We obtain useful special cases of the general induction principle by trivializing one of two codomain families. If $\frown$ is the constant family returning the unit type, we obtain an induction principle for $\mathbb{R}$ alone. If A is constant at the unit type, we obtain an induction principle for the closeness relation $\sim$.

Definition 2.1.3 (The Univalent Foundations Program [1, § 11.3.2])**.** By taking the family $\frown$ in Definition 2.1.2 to be constant at $\mathbf{1}$, we obtain a special case of the general induction principle referred to as **$\mathbb{R}$-induction**. Thus, to construct a section of a type family $A : \mathbb{R} \to \mathcal{U}$, it suffices to specify:

- For each $q : \mathbb{Q}$, an element $f_{\mathsf{rational}}(q) : A(\mathsf{rational}(q))$.
- For each Cauchy approximation $x : \mathbb{Q}_+ \to \mathbb{R}$ and each dependent function $a : \prod_{(\varepsilon:\mathbb{Q}_+)} A(x_\varepsilon)$, an element $f_{\mathsf{limit}}(x,a) : A(\mathsf{limit}(x))$.
- For all $u, v : \mathbb{R}$ such that $u \sim_\varepsilon v$ for all $\varepsilon : \mathbb{Q}_+$, and all $a : A(u)$ and $b : A(v)$, a dependent path $a =^{A}_{\mathsf{path}(u,v)} b$.

Under these hypotheses, $\mathbb{R}$-induction yields a function

$$\mathsf{induction}^{\mathbb{R}} : \prod_{(u:\mathbb{R})} A(u)$$

which satisfies the same computation rules as $(\mathbb{R}, \sim)$-induction. Since $\frown$ is constant at $\mathbf{1}$, proofs for the dependent Cauchy approximation condition are trivially satisfied.

Definition 2.1.4 (The Univalent Foundations Program [1, § 11.3.2])**.** By taking the family A in Definition 2.1.2 to be constant at $\mathbf{1}$, we obtain a special case of the general induction principle referred to as **$\sim$-induction**. Thus, to construct a section of a type family

$$\frown : \prod_{(u,v:\mathbb{R})} \prod_{(\varepsilon:\mathbb{Q}_+)} u \sim_\varepsilon v \to \mathcal{U}$$

it suffices to specify data corresponding to the constructors of $\sim$, where q, r are rationals, $\varepsilon, \delta, \eta$ are positive rationals, and x, y are Cauchy approximations:

- If $-\varepsilon < q - r < \varepsilon$ then $\mathsf{rational}(q) \frown_\varepsilon \mathsf{rational}(r)$.
- If $\varepsilon - \delta > 0$ and $\mathsf{rational}(q) \sim_{\varepsilon-\delta} y_\delta$ then $\mathsf{rational}(q) \frown_\varepsilon \mathsf{limit}(y)$.
- If $\varepsilon - \delta > 0$ and $x_\delta \sim_{\varepsilon-\delta} \mathsf{rational}(r)$ then $\mathsf{limit}(x) \frown_\varepsilon \mathsf{rational}(r)$.
- If $\varepsilon - (\delta + \eta) > 0$, $x_\delta \sim_{\varepsilon-(\delta+\eta)} y_\eta$, and $x_\delta \frown_{\varepsilon-(\delta+\eta)}$ then $\mathsf{limit}(x) \frown_\varepsilon \mathsf{limit}(y)$.
- For all $u, v : \mathbb{R}$, all $\varepsilon : \mathbb{Q}_+$, and all $\varphi : u \sim_\varepsilon v$, the type $u \frown_\varepsilon^\varphi v$ is a proposition.

Under these hypotheses, $\sim$-induction yields a function

$$\mathsf{induction}^{\sim} : \prod_{(u,v:\mathbb{R})} \prod_{(\varepsilon:\mathbb{Q}_+)} u \sim_\varepsilon v \to u \frown_\varepsilon v.$$

Note, since A is constant at $\mathbf{1}$ and $\sim$ takes values in propositions, there is no meaningful dependence on the endpoints in the codomain or the proof of closeness in the domain.

The principle of recursion obtained by making the general induction principle non-dependent is not as useful as we would like. This recursion principle, which is referred to as "ordinary" $(\mathbb{R}, \sim)$-recursion in [1], states that to construct a function $f : \mathbb{R} \to A$, it suffices to provide:

- For every $q : \mathbb{Q}$, an element $f(\mathsf{rational}(q)) : A$.
- For every Cauchy approximation $x : \mathbb{Q}_+ \to \mathbb{R}$, an element $f(\mathsf{limit}(x)) : A$, assuming f has been defined on x_ε for all rational $\varepsilon > 0$.
- For every $u, v : \mathbb{R}$ such that $u \sim_\varepsilon v$ for all rational $\varepsilon > 0$, a proof that $f(u) = f(v)$.

As explained in [1], the last condition is generally difficult to prove unless we have extra information specifying how f behaves on ε-close reals. In other words, we need a way to measure approximate closeness between the images of ε-close reals in the codomain. This is exactly the role played by the family $\frown$ in the general induction principle, but since A is now non-dependent, $\frown$ is simply a type family of the form $A \to A \to \mathbb{Q}_+ \to \mathcal{U}$. Accordingly, [1] introduces a recursion principle whose hypotheses include exactly this extra information.

Definition 2.1.5 (The Univalent Foundations Program [1, § 11.3.2])**.** Let A be a type and let $a \frown_\varepsilon b$ be a type family indexed by $a, b : A$ and $\varepsilon : \mathbb{Q}_+$. The **enhanced principle of** $(\mathbb{R}, \sim)$**-recursion**, which we will refer to simply as $(\mathbb{R}, \sim)$**-recursion** since it is the only recursion principle used here, states that to construct a function $\mathbb{R} \to A$, it suffices to provide the following data:

- A function $f_{\mathsf{rational}} : \mathbb{Q} \to A$.

- For every Cauchy approximation $x : \mathbb{Q}_+ \to \mathbb{R}$ and every map $f' : \mathbb{Q}_+ \to A$ such that

$$f'(\varepsilon) \frown_{\varepsilon+\delta} f'(\delta)$$

for all rational $\varepsilon, \delta > 0$, an element $f_{\mathsf{limit}}(x, f') : A$. Note that this condition amounts to asserting that f' is a dependent Cauchy approximation over x, albeit with the dependence removed, since A is no longer a type family over $\mathbb{R}$. In [1], this is referred to as "a Cauchy approximation with respect to $\frown$". As with the general induction principle, we suppress the relevant witnesses when they are clear from context; otherwise we write $f_{\mathsf{limit}}(x, \varphi, f', \psi)$ for witnesses φ and ψ.

- For all $a, b : A$, if $a \frown_\varepsilon b$ for all $\varepsilon : \mathbb{Q}_+$, then $a = b$. This is referred to as the *separatedness* condition for $\frown$.

- For all $a, b : A$ and all $\varepsilon : \mathbb{Q}_+$, the type $a \frown_\varepsilon b$ is a proposition; that is, the family $\frown$ is a mere relation.

- For all $q, r : \mathbb{Q}$ and all $\varepsilon : \mathbb{Q}_+$, if $-\varepsilon < q - r < \varepsilon$ then $f_{\mathsf{rational}}(q) \frown_\varepsilon f_{\mathsf{rational}}(r)$.

- For all $q : \mathbb{Q}$, all $\varepsilon, \delta : \mathbb{Q}_+$ with $\varepsilon - \delta > 0$, all Cauchy approximations $y : \mathbb{Q}_+ \to \mathbb{R}$, and all Cauchy approximations $g' : \mathbb{Q}_+ \to A$ with respect to $\frown$, if $\mathsf{rational}(q) \sim_{\varepsilon-\delta} y_\delta$ and $f_{\mathsf{rational}}(q) \frown_{\varepsilon-\delta} g'(\delta)$ then

$$f_{\mathsf{rational}}(q) \frown_\varepsilon f_{\mathsf{limit}}(y, g').$$

- For all Cauchy approximations $x : \mathbb{Q}_+ \to \mathbb{R}$, all Cauchy approximations $f' : \mathbb{Q}_+ \to A$

with respect to $\frown$, all $r : \mathbb{Q}$, and all $\varepsilon, \delta : \mathbb{Q}_+$ with $\varepsilon - \delta > 0$, if $x_\delta \sim_{\varepsilon-\delta} \mathsf{rational}(r)$ and $f'(\delta) \frown_{\varepsilon-\delta} f_{\mathsf{rational}}(r)$ then

$$f_{\mathsf{limit}}(x, f') \frown_\varepsilon f_{\mathsf{rational}}(r).$$

- For all Cauchy approximations $x, y : \mathbb{Q}_+ \to \mathbb{R}$, all Cauchy approximations $f', g' : \mathbb{Q}_+ \to A$ with respect to $\frown$, and all $\varepsilon, \delta, \eta : \mathbb{Q}_+$ with $\varepsilon - (\delta + \eta) > 0$, if $x_\delta \sim_{\varepsilon-(\delta+\eta)} y_\eta$ and $f'(\delta) \frown_{\varepsilon-(\delta+\eta)} g'(\eta)$ then

$$f_{\mathsf{limit}}(x, f') \frown_\varepsilon f_{\mathsf{limit}}(y, g').$$

Under these hypotheses, $(\mathbb{R}, \sim)$-recursion yields functions

$$\begin{aligned} &\mathsf{recursion}_{\mathbb{R}} : \mathbb{R} \to A, \\ &\mathsf{recursion}_{\sim} : \prod_{(u,v:\mathbb{R})} \prod_{(\varepsilon:\mathbb{Q}_+)} u \sim_\varepsilon v \to \mathsf{recursion}_{\mathbb{R}}(u) \frown_\varepsilon \mathsf{recursion}_{\mathbb{R}}(v) \end{aligned}$$

such that for all $q : \mathbb{Q}$ and all $x : \mathbb{Q}_+ \to \mathbb{R}$ equipped with a witness φ that x is a Cauchy approximation, the computation rules

$$\begin{aligned} \mathsf{recursion}_{\mathbb{R}}(\mathsf{rational}(q)) &= f_{\mathsf{rational}}(q), \\ \mathsf{recursion}_{\mathbb{R}}(\mathsf{limit}(x, \varphi)) &= f_{\mathsf{limit}}(x, \varphi, \varepsilon \mapsto \mathsf{recursion}_{\mathbb{R}}(x_\varepsilon), \psi) \end{aligned}$$

are satisfied, where for all $\varepsilon, \delta : \mathbb{Q}_+$,

$$\psi(\varepsilon, \delta) := \mathsf{recursion}_\sim(x_\varepsilon, x_\delta, \varepsilon + \delta, \varphi(\varepsilon, \delta))$$

witnesses that the map $\varepsilon \mapsto \mathsf{recursion}_\mathbb{R}(x_\varepsilon)$ is a Cauchy approximation with respect to $\frown$.

2.2 - CLOSENESS

The induction and recursion principles allow us to build up a collection of basic properties about the closeness relation. In turn, these properties make closeness the main tool for constructing functions on the reals, and ultimately, for equipping $\mathbb{R}$ with its algebraic and order-theoretic structure. For example, $\mathbb{R}$-induction shows that ε-closeness is reflexive.

Lemma 2.2.1 (The Univalent Foundations Program [1, Lemma 11.3.8])**.** *For all rational $\varepsilon > 0$, we have $u \sim_\varepsilon u$. In other words, for each rational $\varepsilon > 0$, the binary relation $\sim_\varepsilon$ is reflexive.*

Combining the previous lemma with the fact that the family of closeness relations is separated and takes values in propositions, we can apply Theorem 7.2.2 of [1], which states that a type equipped with a reflexive mere relation implying identity is a set.

Corollary 2.2.2 (The Univalent Foundations Program [1, Theorem 11.3.9])**.** *The HoTT book reals form a set.*

Similarly, $\sim$-induction shows that ε-closeness is symmetric.

Lemma 2.2.3 (The Univalent Foundations Program [1, Lemma 11.3.12])**.** *For all rational $\varepsilon > 0$, if $u \sim_\varepsilon v$ then $v \sim_\varepsilon u$, that is, the relation $\sim_\varepsilon$ is symmetric.*

We also need closeness to satisfy analogues of familiar metric properties. The first is a version of the triangle inequality formulated for relations indexed by a positive rational.

Definition 2.2.1 (The Univalent Foundations Program [1, § 11.3.2])**.** A relation

$$\approx : \mathbb{R} \to \mathbb{R} \to \mathbb{Q}_+ \to \mathcal{U}$$

satisfies the **triangle inequality** if, for all $u, v, w : \mathbb{R}$ and all $\varepsilon, \delta : \mathbb{Q}_+$, if $u \approx_\varepsilon v$ and $v \approx_\delta w$ then $u \approx_{\varepsilon+\delta} w$.

The second condition we need is roundedness.

Definition 2.2.2 (The Univalent Foundations Program [1, § 11.3.2])**.** A relation

$$\approx : \mathbb{R} \to \mathbb{R} \to \mathbb{Q}_+ \to \mathcal{U}$$

is **rounded** if, for all $u, v : \mathbb{R}$ and all $\varepsilon : \mathbb{Q}_+$,

$$u \approx_\varepsilon v \iff \exists \theta : \mathbb{Q}_+, (\theta < \varepsilon) \times (u \approx_{\varepsilon-\theta} v).$$

The implication from left to right is called **openness**, and the converse implication is called **monotonicity**.

At first this openness condition may seem counterintuitive: why should ε-closeness imply $(\varepsilon - \theta)$-closeness for some strictly positive θ? The point is that $u \sim_\varepsilon v$ is intended to express a *strict* inequality. Indeed, once the algebraic and order structure of $\mathbb{R}$ has been developed, Theorem 11.3.44 of [1] shows

$$(u \sim_\varepsilon v) \simeq (|u - v| < \mathsf{rational}(\varepsilon))\,.$$

From this perspective, the openness condition becomes much more natural.

This phenomenon is visible in the rational case. We can show that the relation $\sim : \mathbb{Q} \to \mathbb{Q} \to \mathbb{Q}_+ \to \mathcal{U}$ given by

$$q \sim_\varepsilon r := |q - r| < \varepsilon$$

satisfies the openness condition.

Lemma 2.2.4. *The rational closeness relation is open.*

Proof. Suppose $|q - r| < \varepsilon$. Define

$$\theta := \frac{\varepsilon - |q - r|}{2}.$$

Then $\varepsilon - |q - r| > 0$, so $0 < \theta$. Moreover,

$$\varepsilon - \theta = \varepsilon - \frac{\varepsilon - |q - r|}{2} = \frac{|q - r| + \varepsilon}{2}$$

which is the midpoint between $|q - r|$ and ε, and since $|q - r| < \varepsilon$, it lies strictly between $|q - r|$ and ε. In particular,

$$|q - r| < \varepsilon - \theta.$$

Hence $q \sim_{\varepsilon - \theta} r$. □

However, reasoning with midpoints in this manner assumes that closeness is induced by a distance metric. At this stage, the closeness relation has only been given inductively, so we do not yet have the characterization of Theorem 11.3.44 available. Since the characterization itself depends on the triangle inequality and roundedness, appealing to it here would be circular. We therefore need a different way to analyze the inductively defined closeness relation.

There is a second difficulty. Definition 2.1.1 defines the closeness relation by constructors, so its clauses tell us how to *construct* witnesses of ε-closeness. For example, the rational-rational constructor gives a map

$$-\varepsilon < q - r < \varepsilon \to \mathsf{rational}(q) \sim_{\varepsilon} \mathsf{rational}(r).$$

Thus the displayed inequality is sufficient to produce a proof that $\mathsf{rational}(q)$ and $\mathsf{rational}(r)$ are ε-close.

What is missing is the converse implication. If we are instead given a proof of $\mathsf{rational}(q) \sim_{\varepsilon} \mathsf{rational}(r)$ as a hypothesis, the inductive definition does not by itself allow us to conclude that $-\varepsilon < q - r < \varepsilon$. The same issue arises in the other three cases involving limits. In other words, the constructor clauses for closeness do not yet provide a case-by-case characterization of closeness when it appears as an assumption.

The strategy in [1] is therefore to define an auxiliary closeness relation $\approx$ by recursion, so that when its arguments are rational points or limits, the expression $u \approx_{\varepsilon} v$ unfolds according to one of four explicit clauses. In this sense, $\approx$ computes on the point constructors of the reals. This gives a closeness relation whose behavior can be read off immediately from the constructors used in its arguments.

Theorem 2.2.5 (The Univalent Foundations Program [1, Theorem 11.3.16])**.** *There is a family of mere relations* $\approx: \mathbb{R} \to \mathbb{R} \to \mathbb{Q}_+ \to \mathcal{U}$ *such that*

$$(\mathsf{rational}(q) \approx_\varepsilon \mathsf{rational}(r)) := (-\varepsilon < q - r < \varepsilon)$$

$$(\mathsf{rational}(q) \approx_\varepsilon \mathsf{limit}(y)) := \exists \delta : \mathbb{Q}_+, \mathsf{rational}(q) \approx_{\varepsilon-\delta} y_\delta$$

$$(\mathsf{limit}(x) \approx_\varepsilon \mathsf{rational}(r)) := \exists \delta : \mathbb{Q}_+, x_\delta \approx_{\varepsilon-\delta} \mathsf{rational}(r)$$

$$(\mathsf{limit}(x) \approx_\varepsilon \mathsf{limit}(y)) := \exists \delta, \eta : \mathbb{Q}_+, x_\delta \approx_{\varepsilon-(\delta+\eta)} y_\eta.$$

Moreover, $\approx$ *is rounded and satisfies the mixed triangle laws*

$$(u \approx_\varepsilon v) \to (v \sim_\delta w) \to (u \approx_{\varepsilon+\delta} w),$$

$$(u \sim_\varepsilon v) \to (v \approx_\delta w) \to (u \approx_{\varepsilon+\delta} w).$$

The advantage of $\approx$ is that these explicit clauses provide the converse information that was missing from $\sim$. To use this information for the original closeness relation, however, we need to show that the two relations coincide.

Theorem 2.2.6 (The Univalent Foundations Program [1, Theorem 11.3.32])**.** *For any* $u, v : \mathbb{R}$ *and* $\varepsilon : \mathbb{Q}_+$*, we have*

$$u \sim_\varepsilon v \iff u \approx_\varepsilon v.$$

Since both sides are mere propositions, it follows that

$$(u \sim_\varepsilon v) = (u \approx_\varepsilon v).$$

Once this equivalence is established, the explicit characterization of $\approx$ can be transferred back to $\sim$, along with the roundedness and triangle properties built into the construction of $\approx$.

Corollary 2.2.7 (The Univalent Foundations Program [1, Corollary 11.3.33])**.** *Closeness is rounded and satisfies the triangle inequality.*

The results of this section supply the basic structural properties of the closeness relation needed in the remainder of the development. In particular, roundedness and the triangle inequality make closeness workable as a substitute for metric reasoning, while the auxiliary relation $\approx$ provides the explicit case analysis needed to extract usable information from closeness hypotheses. We next use these properties to provide lifts of Lipschitz and non-expanding maps from the rationals to the reals.

2.3 - LIFTING LIPSCHITZ AND NONEXPANDING MAPS

One conceptual advantage of the HoTT book reals is that their construction forces us to make explicit the principle by which maps on the rationals extend to the reals. In the classical quotient construction of the Cauchy reals, we typically define operations on representatives and then prove that they are well-defined on equivalence classes. This procedure relies on the fact that a suitably continuous map on the dense subspace $\mathbb{Q}$ extends to the completion $\mathbb{R}$, but the corresponding well-definedness argument can easily recede into the background as a routine technical step. In the higher-inductive setting, however, there are no such representatives, so we must formulate and prove extension principles explicitly using the recursion principle for the reals.

The HoTT book develops extension lemmas for Lipschitz and non-expanding maps [1]. These hypotheses are strong enough to lift important operations such as addition and min/max, but they do not capture every operation we ultimately want to define on the reals. In particular, neither multiplication nor reciprocal are globally Lipschitz, so their definitions cannot be obtained from the following lemmas alone and require additional work in Section 2.4. Accordingly, the results of this section should be read as a collection of extension principles tailored to the later algebraic development, rather than as a final characterization of the conditions under which maps can be lifted from $\mathbb{Q}$ to $\mathbb{R}$. A broader extension theorem, for example for uniformly continuous maps, would be desirable but is beyond the scope of this thesis; see Chapter 5.

The first extension principle concerns Lipschitz maps, whose defining condition is naturally expressed in terms of the closeness relation.

Definition 2.3.1 (The Univalent Foundations Program [1, Definition 11.3.14])**.** In the rational case, a map $f : \mathbb{Q} \to \mathbb{R}$ is **Lipschitz** if it comes equipped with an element $L : \mathbb{Q}_+$, the **Lipschitz** constant, such that, for all $q, r : \mathbb{Q}$ and $\varepsilon : \mathbb{Q}_+$,

$$|q - r| < \varepsilon \implies f(q) \sim_{L\varepsilon} f(r).$$

Analogously, for the real case, a map $g : \mathbb{R} \to \mathbb{R}$ is **Lipschitz** if it comes equipped with an element $L : \mathbb{Q}_+$ such that, for all $u, v : \mathbb{R}$ and $\varepsilon : \mathbb{Q}_+$,

$$u \sim_\varepsilon v \implies g(u) \sim_{L\varepsilon} g(v).$$

With this definition in place, we can state the extension principle.

Lemma 2.3.1 (The Univalent Foundations Program [1, Lemma 11.3.15])**.** *Let $f : \mathbb{Q} \to \mathbb{R}$ be Lipschitz with constant $L : \mathbb{Q}_+$. Then there is a map $\overline{f} : \mathbb{R} \to \mathbb{R}$ which is Lipschitz with the same constant L and satisfies*

$$\overline{f}(\mathsf{rational}(q)) = f(q)$$

for all $q : \mathbb{Q}$.

For the binary operations needed later, [1] does not state a fully general two-variable Lipschitz extension lemma. Instead, it isolates the special case of maps that are non-expanding, that is, Lipschitz with constant $L \leq 1$ in each variable separately. This hypothesis is more restrictive than allowing an arbitrary Lipschitz constant, but apart from multiplication and reciprocal it covers the operations we want to extend in Section 2.4.

Definition 2.3.2 (The Univalent Foundations Program [1, Lemma 11.3.40])**.** A map $f : \mathbb{Q} \to \mathbb{Q}$ is **non-expanding** if, for all $q, r : \mathbb{Q}$,

$$|f(q) - f(r)| \leq |q - r|.$$

Equivalently, if $|q - r| < \varepsilon$, then $|f(q) - f(r)| < \varepsilon$ for every $\varepsilon : \mathbb{Q}_+$. This is the form that generalizes directly to the reals: a map $f : \mathbb{R} \to \mathbb{R}$ is non-expanding if, for all $u, v : \mathbb{R}$ and $\varepsilon : \mathbb{Q}_+$,

$$u \sim_\varepsilon v \implies f(u) \sim_\varepsilon f(v).$$

Moreover, a function $\mathbb{Q} \to \mathbb{Q} \to \mathbb{Q}$ is non-expanding if, for all $r : \mathbb{Q}$, the map $f(-, r)$ is non-expanding and, for all $q : \mathbb{Q}$, the map $f(q, -)$ is non-expanding. The two-variable real case is defined analogously.

Lemma 2.3.2 (The Univalent Foundations Program [1, Lemma 11.3.40])**.** *Let $f : \mathbb{Q} \to \mathbb{Q} \to \mathbb{Q}$ be a non-expanding map. Then there is a non-expanding map $\overline{f} : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ such that*

$$\overline{f}(\mathsf{rational}(q), \mathsf{rational}(r)) = \mathsf{rational}(f(q, r)).$$

for all $q, r : \mathbb{R}$.

In addition to proving that such extensions exist, we also need to know when they are uniquely determined. The relevant uniqueness statement can be more generally formulated in terms of arbitrary continuous functions, so we first express the standard definition of continuity in terms of the closeness relation.

Definition 2.3.3 (The Univalent Foundations Program [1, Lemma 11.3.39])**.** A map $f : \mathbb{R} \to \mathbb{R}$ is **continuous at a point** $u : \mathbb{R}$ if for all $\varepsilon : \mathbb{Q}_+$, there merely exists a $\delta : \mathbb{Q}_+$ such that for all $v : \mathbb{R}$,

$$u \sim_\delta v \implies f(u) \sim_\varepsilon f(v).$$

A map $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if it is continuous for every point $u : \mathbb{R}$.

This definition encompasses the extensions constructed above. Indeed, any Lipschitz map is continuous: given ε, choose $\delta := \varepsilon/L$; then $u \sim_\delta v$ implies $f(u) \sim_\varepsilon f(v)$. As the special case of a Lipschitz map with constant $L \leq 1$, every univariate non-expanding map is continuous. Likewise, if $f : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ is non-expanding in each variable separately, then it will be continuous in each variable separately.

Lemma 2.3.3 (The Univalent Foundations Program [1, Lemma 11.3.39]). *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. If $f \circ \mathsf{rational} = g \circ \mathsf{rational}$ then $f = g$.*

The previous lemma gives us a way to extend identities from the rationals to identities on the reals. For example, once addition on $\mathbb{R}$ has been constructed, the maps

$$u \mapsto u + 0 \qquad \text{and} \qquad u \mapsto u$$

are continuous and agree on rational inputs, since $q + 0 = q$ for all $q : \mathbb{Q}$. The previous lemma therefore yields the right unit law for real addition:

$$u + 0 = u$$

for all $u : \mathbb{R}$.

The authors of [1] indicate that identities in several variables can be extended in the same manner. However, the precise hypotheses needed to make this work are not completely obvious. In the binary case, for example, to prove commutativity of addition, should we require addition to be jointly continuous as a function

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

or does continuity in each variable separately suffice? At first glance, the latter condition may seem too weak. In fact, separate continuity is sufficient, because the one-variable uniqueness lemma can be applied one coordinate at a time. Since this result is used repeatedly later and is not stated explicitly in [1], we formulate and prove it here.

Lemma 2.3.4. *Let $f, g : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ be maps that are continuous in both variables separately. That is, for every $u : \mathbb{R}$, the maps*

$$f(u, -), g(u, -) : \mathbb{R} \to \mathbb{R}$$

are continuous, and for every $v : \mathbb{R}$, the maps

$$f(-, v), g(-, v) : \mathbb{R} \to \mathbb{R}$$

are continuous. Suppose that

$$f(\mathsf{rational}(q), \mathsf{rational}(r)) = g(\mathsf{rational}(q), \mathsf{rational}(r))$$

for all $q, r : \mathbb{Q}$. Then

$$f(u, v) = g(u, v)$$

for all $u, v : \mathbb{R}$*, and hence* $f = g$*.*

Proof. Fix $u : \mathbb{R}$. We show that

$$f(u, v) = g(u, v)$$

for all $v : \mathbb{R}$.

By the assumed continuity of $f(u, -)$ and $g(u, -)$, the maps

$$v \mapsto f(u, v) \qquad \text{and} \qquad v \mapsto g(u, v)$$

are continuous. Thus, by Lemma 2.3.3, it suffices to prove that

$$f(u, \mathsf{rational}(r)) = g(u, \mathsf{rational}(r))$$

for all $r : \mathbb{Q}$.

Fix $r : \mathbb{Q}$. By the assumed continuity of $f(-, \mathsf{rational}(r))$ and $g(-, \mathsf{rational}(r))$, the maps

$$w \mapsto f(w, \mathsf{rational}(r)) \qquad \text{and} \qquad w \mapsto g(w, \mathsf{rational}(r))$$

are continuous. Moreover, for every $q : \mathbb{Q}$, the hypothesis gives

$$f(\mathsf{rational}(q), \mathsf{rational}(r)) = g(\mathsf{rational}(q), \mathsf{rational}(r)).$$

Applying Lemma 2.3.3, we conclude that

$$f(w, \mathsf{rational}(r)) = g(w, \mathsf{rational}(r))$$

for all $w : \mathbb{R}$. In particular, evaluating at $w = u$ yields

$$f(u, \mathsf{rational}(r)) = g(u, \mathsf{rational}(r)).$$

Since $r : \mathbb{Q}$ was arbitrary, the required agreement on rational inputs follows, and therefore

$$f(u, v) = g(u, v)$$

for all $v : \mathbb{R}$. Since $u : \mathbb{R}$ was arbitrary, it follows that

$$f(u, v) = g(u, v)$$

for all $u, v : \mathbb{R}$, and hence $f = g$. □

The previous lemma can be packaged as a convenient extension law for binary operations. This is the form that used repeatedly in Section 2.4 to extend identities from the rationals to the reals.

Corollary 2.3.5. *Let* $f, g : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ *and* $f', g' : \mathbb{Q} \to \mathbb{Q} \to \mathbb{Q}$*. Suppose that*

$$f(\mathsf{rational}(q), \mathsf{rational}(r)) = \mathsf{rational}(f'(q, r))$$

and

$$g(\mathsf{rational}(q), \mathsf{rational}(r)) = \mathsf{rational}(g'(q, r))$$

for all $q, r : \mathbb{Q}$*, and that*

$$f'(q, r) = g'(q, r)$$

for all $q, r : \mathbb{Q}$*. Assume moreover that* f *and* g *are continuous in each variable separately. Then*

$$f(u, v) = g(u, v)$$

for all $u, v : \mathbb{R}$*, and hence* $f = g$*.*

Proof. For any $q, r : \mathbb{Q}$, the assumptions give a path

$$f(\mathsf{rational}(q), \mathsf{rational}(r)) = \mathsf{rational}(f'(q, r)) = \mathsf{rational}(g'(q, r)) = g(\mathsf{rational}(q), \mathsf{rational}(r)).$$

Thus f and g agree on rational pairs. Since both maps are continuous in both variables separately, the result follows by Lemma 2.3.4. □

The same pattern applies to functions of three or more variables: we extend agreement one coordinate at a time, using separate continuity in the relevant variable at each step. We do not formulate a general n-ary version here, since in Section 2.4 we only extend identities involving at most three variables, and a fully general statement would require additional bookkeeping for uncurried n-ary functions.

This completes the extension principles needed in the next section. The lifting lemmas provide operations on $\mathbb{R}$, while the uniqueness lemmas explain precisely how identities on $\mathbb{Q}$

induce identities on $\mathbb{R}$.

2.4 - ALGEBRA AND ORDER

The previous three sections provide the tools needed to equip the HoTT book reals with their algebraic and order-theoretic structure. Because the corresponding Agda development for algebra and order spans more than six thousand lines, we do not reproduce it result by result. Instead, we organize this section around three case studies showing how the tools from the previous sections are used in practice. First, we lift negation, addition, minimum, and maximum from the rationals and illustrate how algebraic identifiers transfer from $\mathbb{Q}$ to $\mathbb{R}$; this also yields the induced non-strict order. Second, we define strict order and develop the lemmas needed to show that addition preserves and reflects it. Third, we construct multiplication and reciprocal, which require extra work beyond the extension principles of Section 2.3.

This organization also reflects the division of labor among our sources. For the first study, we follow the HoTT book closely. For the development of strict inequality, and especially for the proof that addition preserves and reflects strict inequality, we follow Gilbert [3], since the HoTT book does not provide that argument. For multiplication and reciprocal, we likewise depart from the HoTT book's squaring-based construction and instead adopt Gilbert's more direct construction using iterated Lipschitz extension. The section closes by assembling the results into the statement that the HoTT book reals form an Archimedean ordered field.

We begin with the first case study, concerning the operations obtained directly by lifting from the rationals. Negation is obtained by Lipschitz extension, while addition and min/max are obtained by non-expanding extension. These constructions all follow a common pattern. First, we prove that the rational version of the operation is Lipschitz or non-expanding.

We then use this result to extend the operation to the reals by applying Lemma 2.3.1 or Lemma 2.3.2. In this way we obtain operations

$$- : \mathbb{R} \to \mathbb{R}, \qquad + : \mathbb{R} \to \mathbb{R} \to \mathbb{R}, \qquad \min, \max : \mathbb{R} \to \mathbb{R} \to \mathbb{R}.$$

Then, for each basic algebraic law the operation should satisfy, we use or prove the corresponding result on the rationals and apply Corollary 2.3.5 and its analogues for functions of higher arity.

Here, we walk through the construction of the max operation and prove that it satisfies

$$\max(a + u, a + v) = a + \max(u, v) \tag{2.2}$$

for all $a, u, v : \mathbb{R}$. Once the order relation has been defined, this identity will be the key to showing that addition preserves and reflects order, that is, that we have

$$u \leq v \iff a + u \leq a + v$$

for all $a, u, v : \mathbb{R}$. In addition to illustrating the common pattern well, this example is also not covered explicitly in either [1] or [3].

We begin by showing that the max operation on rationals is non-expanding in both arguments separately. To do so, we first record a convenient closed-form expression for $\max(q, r)$ in terms of addition, subtraction, and absolute value. This identity will be used twice: first to establish the required non-expandingness of rational max, and then to prove its translation invariance on the rationals, which will later be lifted to the reals.

Lemma 2.4.1. *For all rational $q, r : \mathbb{Q}$, we have*

$$\max(q, r) = \frac{q + r}{2} + \frac{|q - r|}{2}.$$

Proof. This follows by cases on $q \leq r$ and $r \leq q$. □

We first use this identity to prove that rational max is non-expanding in each argument.

Lemma 2.4.2. *The max operation on the rationals is non-expanding in both variables separately.*

Proof. Fix $q, r, s : \mathbb{Q}$. We show

$$|\max(q, s) - \max(r, s)| \leq |q - r|, \tag{2.3}$$

$$|\max(q, r) - \max(q, s)| \leq |r - s|. \tag{2.4}$$

It suffices to prove eq. (2.3), since eq. (2.4) then follows from the commutativity of max and the symmetry of distance. We have

$$\begin{aligned}
|\max(q, s) - \max(r, s)| &= \left| \frac{(q + s) + |q - s|}{2} - \frac{(r + s) + |r - s|}{2} \right| && \text{by Lemma 2.4.1} \\
&= \frac{1}{2} \left| (q - r) + (|q - s| - |r - s|) \right| \\
&\leq \frac{1}{2} |q - r| + \frac{1}{2} \left| |q - s| - |r - s| \right| && \text{by the triangle inequality} \\
&\leq \frac{1}{2} |q - r| + \frac{1}{2} |(q - s) - (r - s)| && \text{by the reverse triangle inequality} \\
&= \frac{1}{2} |q - r| + \frac{1}{2} |q - r| \\
&= |q - r|,
\end{aligned}$$

which proves eq. (2.3), and hence the lemma. □

Applying the binary non-expanding extension principle (Lemma 2.3.2) to the previous lemma, we obtain a function

$$\max : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$$

extending rational max. Since the extension is again non-expanding in each variable separately, max is continuous in each variable separately as well. To illustrate how identities on $\mathbb{Q}$ are transferred to $\mathbb{R}$, we now prove that rational max is translation invariant and then lift that identity to the reals.

Lemma 2.4.3. *For all rational $q, r, a : \mathbb{Q}$, we have*

$$\max(a + q, a + r) = a + \max(q, r).$$

Proof. Using Lemma 2.4.1, we compute

$$\begin{aligned}
\max(a + q, a + r) &= \frac{(a + q) + (a + r)}{2} + \frac{|(a + q) - (a + r)|}{2} \\
&= a + \frac{q + r}{2} + \frac{|q - r|}{2} \\
&= a + \max(q, r),
\end{aligned}$$

as required. □

To lift Lemma 2.4.3 to the reals, we must show that the two sides of the desired identity define separately continuous maps of three real variables. Most of these continuity claims follow directly from the fact that addition and max are non-expanding in each argument separately. The only slightly nontrivial case is the dependence on the translation parameter a in expressions such as

$$a \mapsto \max(a + u, a + v).$$

This is a composite of two Lipschitz maps with the binary operation max, and similar combinations arise repeatedly throughout the formalization. We therefore record the following binary composition lemma once and use it here as a representative instance.

Lemma 2.4.4. *Let $f, g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$. Suppose that f is Lipschitz with constant L, g is Lipschitz with constant M, and for every $v : \mathbb{R}$ the map $u \mapsto h(u, v)$ is Lipschitz with constant N_1, while for every $u : \mathbb{R}$ the map $v \mapsto h(u, v)$ is Lipschitz with constant N_2. Then the composite*

$$u \mapsto h(f(u), g(u))$$

is Lipschitz with constant $N_1 L + N_2 M$.

Proof. Let $u, v : \mathbb{R}$, let $\varepsilon : \mathbb{Q}_+$, and suppose that

$$u \sim_\varepsilon v.$$

Since f and g are Lipschitz with constants L and M respectively, we have

$$f(u) \sim_{L\varepsilon} f(v) \qquad \text{and} \qquad g(u) \sim_{M\varepsilon} g(v).$$

Fixing the second argument, the Lipschitz hypothesis on $w \mapsto h(w, g(u))$ gives

$$h(f(u), g(u)) \sim_{N_1 L\varepsilon} h(f(v), g(u)).$$

Similarly, fixing the first argument, the Lipschitz hypothesis on $z \mapsto h(f(v), z)$ yields

$$h(f(v), g(u)) \sim_{N_2 M\varepsilon} h(f(v), g(v)).$$

Applying the triangle inequality for closeness, we obtain

$$h(f(u), g(u)) \sim_{N_1 L\varepsilon + N_2 M\varepsilon} h(f(v), g(v)).$$

Since

$$N_1 L\varepsilon + N_2 M\varepsilon = (N_1 L + N_2 M)\varepsilon,$$

this shows that $u \mapsto h(f(u), g(u))$ is Lipschitz with constant $N_1 L + N_2 M$. □

With this auxiliary lemma in hand, we can now transfer translation invariance of max from the rationals to the reals.

Lemma 2.4.5. *For all real $a, u, v : \mathbb{R}$, we have*

$$\max(a + u, a + v) = a + \max(u, v).$$

Proof. Define maps $f, g : \mathbb{R} \to \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ by

$$f(a, u, v) := \max(a + u, a + v),$$

$$g(a, u, v) := a + \max(u, v).$$

Likewise, define $f', g' : \mathbb{Q} \to \mathbb{Q} \to \mathbb{Q} \to \mathbb{Q}$ by

$$f'(a, q, r) := \max(a + q, a + r),$$

$$g'(a, q, r) := a + \max(q, r).$$

By construction of addition and max on $\mathbb{R}$ as non-expanding lifts of the corresponding rational operations, f and g agree with f' and g' on rational inputs. Lemma 2.4.3 gives

$$f'(a, q, r) = g'(a, q, r)$$

for all $a, q, r : \mathbb{Q}$. It therefore remains to check that f and g are continuous in each variable separately, so that the three-variable extension argument described after Corollary 2.3.5 applies.

For f, the maps

$$u \mapsto \max(a + u, a + v),$$

$$v \mapsto \max(a + u, a + v)$$

are composites of continuous maps, since addition and max are non-expanding (and hence continuous) in both variables separately. For fixed u and v, the maps

$$a \mapsto a + u \qquad \text{and} \qquad a \mapsto a + v$$

are Lipschitz with constant 1, while max is Lipschitz with constant 1 in both variables separately. Hence Lemma 2.4.4 shows that

$$a \mapsto \max(a + u, a + v)$$

is Lipschitz with constant 2, and therefore continuous.

For g, the maps

$$u \mapsto a + \max(u, v),$$
$$v \mapsto a + \max(u, v)$$

are again composites of continuous maps. For fixed u and v, the map

$$a \mapsto a + \max(u, v)$$

is Lipschitz with constant 1, since addition on the left is Lipschitz with constant 1. Thus g is also continuous in each variable separately.

The three-variable analogue of Corollary 2.3.5 therefore yields

$$f(a, u, v) = g(a, u, v)$$

for all $a, u, v : \mathbb{R}$, which is exactly the desired identity. $\square$

The same transfer argument applies to the remaining algebraic laws for addition, negation, min, and max. Since the corresponding identities hold on $\mathbb{Q}$, they lift to $\mathbb{R}$ by the uniqueness of continuous extensions. In particular, addition and negation endow $\mathbb{R}$ with the structure of an Abelian group, while min and max satisfy the usual associative, commutative, idempotent, and absorption laws.

The max operation in turn induces a partial order on $\mathbb{R}$ by defining

$$u \leq v := \max(u, v) = v.$$

Reflexivity, antisymmetry, and transitivity then follow from idempotence, commutativity, and associativity of max, respectively. With the definition of $\leq$ in place, Lemma 2.4.5 has the expected consequence that addition preserves and reflects order.

Lemma 2.4.6. *For each real $a : \mathbb{R}$, the maps*

$$u \mapsto a + u \qquad \textit{and} \qquad u \mapsto u + a$$

are monotone and order-reflecting with respect to $\leq$. Explicitly,

$$u \leq v \iff a + u \leq a + v$$

and hence also

$$u \leq v \iff u + a \leq v + a$$

for all $a, u, v : \mathbb{R}$.

Proof. By commutativity of addition, it suffices to prove the claim for addition on the left.

($\Longrightarrow$) Suppose $u \leq v$. Then by definition, $\max(u, v) = v$. It follows by Lemma 2.4.5 that

$$\max(a + u, a + v) = a + \max(u, v) = a + v,$$

and hence $a + u \leq a + v$, again by the definition of $\leq$.

($\Longleftarrow$) Suppose $a + u \leq a + v$. Then by definition

$$\max(a + u, a + v) = a + v.$$

Applying Lemma 2.4.5 with $-a$ in place of a, we have

$$\begin{aligned}
\max(u, v) &= \max((-a) + (a + u), (-a) + (a + v)) \\
&= (-a) + \max(a + u, a + v) \\
&= (-a) + (a + v) \\
&= v,
\end{aligned}$$

and so $u \leq v$. □

We now turn to the second case study, concerning strict order. The non-strict order relation $\leq$ was induced from the max operation, lifted from the corresponding operation on rationals. In contrast, strict inequality cannot be obtained directly from the lift of some algebraic operation. As a result, we will not be able to immediately derive properties of the strict order relation as a consequence of the uniqueness of continuous extensions; proving properties of the strict order relation will require a bit more attention. Therefore, our goal in this case study is not only to define $<$, but to develop enough theory around it to use it in later algebraic arguments. The HoTT book proves several lemmas about order and gives a characterization of closeness by distance, but it does not for instance show that addition preserves or reflects strict inequality [1]. For that result, we follow Gilbert [3] in deriving an alternative characterization of strict inequality: namely, that $u < v$ holds exactly when there merely exists a positive rational ε such that

$$u + \mathsf{rational}(\varepsilon) \leq v.$$

The results below are organized around this characterization. We begin with the definition of $<$ and its immediate consequences, then prove perturbation lemmas relating $<$ to closeness, and finally derive Gilbert's characterization to show that addition preserves and reflects strict inequality.

The definition of strict inequality given in [1] makes use of our intuition that the reals should satisfy the Archimedean property. We can utilize our expectation that the rationals are dense in the reals[2] to express strict inequality as

$$u < v :\equiv \exists q, r : \mathbb{Q}, (u \leq \mathsf{rational}(q)) \times (q < r) \times (\mathsf{rational}(r) \leq v)$$

where $q < r$ is strict inequality on the rationals [1, §11.3.3].

[2]This is equivalent to the more typical characterization that for any $u : \mathbb{R}$ there exists an integer k with $u < k$. See [1, Exercise 11.7].

The irreflexivity and transitivity of $<$ follow immediately from showing that the $\mathsf{rational}$ constructor preserves and reflects $\leq$ and then in turn $<$. We also obtain proofs that $\leq$ and $<$ satisfy

$$u < v \to v \leq w \to u < w,$$

$$u \leq v \to v < w \to u < w$$

for all $u, v, w : \mathbb{R}$. As indicated, the Archimedean property follows immediately from the definition.

Theorem 2.4.7 (Archimedean principle; The Univalent Foundations Program [1, Theorem 11.3.41])**.** *For every $u, v : \mathbb{R}$ with $u < v$, there merely exists $q : \mathbb{Q}$ such that $u < \mathsf{rational}(q) < v$.*

Proof. Suppose $u < v$. Then by definition there exist $r, s : \mathbb{Q}$ such that

$$u \leq \mathsf{rational}(r), \qquad r < s, \qquad \mathsf{rational}(s) \leq v.$$

Choose $q := \frac{r+s}{2}$, the midpoint of r and s. Then $r < q < s$.

First, we have

$$u \leq \mathsf{rational}(r), \qquad r < q, \qquad \mathsf{rational}(q) \leq \mathsf{rational}(q),$$

where the last inequality is by reflexivity of $\leq$. Thus we obtain $u < \mathsf{rational}(q)$ by definition.

Similarly, we have

$$\mathsf{rational}(q) \leq \mathsf{rational}(q), \qquad q < s, \qquad \mathsf{rational}(s) \leq v,$$

so again by definition of $<$ we obtain $\mathsf{rational}(q) < v$.

Hence $u < \mathsf{rational}(q) < v$. □

This gives us the immediate consequences of the definition of strict inequality. To make further use of $<$, however, we need to understand how it interacts with the closeness relation. The next two lemmas from the HoTT book describe how small perturbations interact with non-strict and strict rational bounds. They are followed by an important characterization of closeness in terms of distance, confirming that the inductively defined closeness relation $\sim$ is equivalent to the more familiar notion of closeness induced by the distance metric $d(u, v) := |u - v|$ on the reals.

Lemma 2.4.8 (The Univalent Foundations Program [1, Lemma 11.3.42])**.** *Let* $u, v : \mathbb{R}$*,* $q : \mathbb{Q}$ *and suppose* $u \leq \mathsf{rational}(q)$*. If* $u \sim_\varepsilon v$ *for some* $\varepsilon : \mathbb{Q}_+$ *then* $v \leq \mathsf{rational}(q + \varepsilon)$*.*

Lemma 2.4.9 (The Univalent Foundations Program [1, Lemma 11.3.43(i)])**.** *Let* $u, v : \mathbb{R}$*,* $q : \mathbb{Q}$ *and assume* $u < \mathsf{rational}(q)$*. If* $u \sim_\varepsilon v$ *for some* $\varepsilon : \mathbb{Q}_+$ *then* $v < \mathsf{rational}(q + \varepsilon)$*.*

Theorem 2.4.10 (The Univalent Foundations Program [1, Theorem 11.3.44])**.** *For all* $u, v : \mathbb{R}$ *and all* $\varepsilon : \mathbb{Q}_+$*, we have*

$$(u \sim_\varepsilon v) \iff (|u - v| < \mathsf{rational}(\varepsilon)) .$$

Up to this point, we have followed the development for strict order given in [1]. We now turn to Gilbert's strategy for proving that addition preserves and reflects strict inequality. The key idea is to derive a second characterization of $<$ involving addition by a positive rational

$$u < v \iff \exists \varepsilon : \mathbb{Q}_+, u + \mathsf{rational}(\varepsilon) \leq v.$$

This reformulation is useful because it lets us build on the algebraic structure already developed for non-strict inequality. Once $<$ is expressed in terms of addition and $\leq$, we can use the fact that addition already preserves and reflects $\leq$ to prove the corresponding result for strict inequality.

The route to this characterization proceeds in four steps. We first prove a helper lemma showing that ε-closeness yields a non-strict order bound of the form

$$v \leq u + \mathsf{rational}(\varepsilon).$$

From this we derive two perturbation lemmas for strict inequality, showing how an inequality $u < v$ is affected when either endpoint is replaced by a sufficiently close real. We then establish the weak linearity of $<$, which is needed to prove that every real lies strictly below any positive rational perturbation of itself. With these results in place, Gilbert's alternative characterization of strict inequality follows.

We begin with the promised non-strict order bound arising from closeness.

Lemma 2.4.11 (Gilbert [3, Lemma 4.2])**.** *For all $u, v : \mathbb{R}$ and $\varepsilon : \mathbb{Q}_+$, if $u \sim_\varepsilon v$ then $v \leq u + \mathsf{rational}(\varepsilon)$.*

Proof. Suppose $u \sim_\varepsilon v$. By Theorem 2.4.10, this implies

$$|u - v| < \mathsf{rational}(\varepsilon),$$

which we can weaken to

$$|u - v| \leq \mathsf{rational}(\varepsilon).$$

Since $-(u - v) \leq |u - v|$, we have

$$-u + v = -(u - v) \leq |u - v| \leq \mathsf{rational}(\varepsilon).$$

By the monotonicity of addition (Lemma 2.4.6), adding u to both sides yields

$$v \leq u + \mathsf{rational}(\varepsilon)$$

which is the desired inequality. □

The next step is to promote this non-strict control to corresponding perturbation results for strict inequality.

Lemma 2.4.12 (Gilbert [3, Lemma 4.3])**.** *For all $u, v, w : \mathbb{R}$ and $\varepsilon : \mathbb{Q}_+$, if $u < v$ and $u \sim_\varepsilon w$ then $w < v + \mathsf{rational}(\varepsilon)$.*

Proof. Suppose $u < v$ and that $u \sim_\varepsilon w$ for some $\varepsilon : \mathbb{Q}_+$. By the Archimedean property (Theorem 2.4.7), there merely exists some $q : \mathbb{Q}$ such that

$$u < \mathsf{rational}(q) < v.$$

Applying Lemma 2.4.9 to $u < \mathsf{rational}(q)$ and $u \sim_\varepsilon w$, we obtain

$$w < \mathsf{rational}(q + \varepsilon).$$

Weakening $\mathsf{rational}(q) < v$ to $\mathsf{rational}(q) \leq v$ and adding ε to both sides (Lemma 2.4.6) yields

$$\mathsf{rational}(q + \varepsilon) \leq v + \mathsf{rational}(\varepsilon).$$

By the transitivity of $\leq$ and $<$, it follows that

$$w < v + \mathsf{rational}(\varepsilon)$$

as needed. □

Although Gilbert does not formulate the next corollary separately, recording it explicitly makes the subsequent proof of weak linearity a bit cleaner.

Corollary 2.4.13. *For all $u, v, w : \mathbb{R}$ and $\varepsilon : \mathbb{Q}_+$, if $u < v$ and $v \sim_\varepsilon w$ then $u - \mathsf{rational}(\varepsilon) < w$.*

Proof. Assume $u < v$ and that $v \sim_\varepsilon w$ for some $\varepsilon : \mathbb{Q}_+$. Negation is antitone (order reversing) with respect to $<$, so we have $-v < -u$. Negation is also non-expanding, so from $v \sim_\varepsilon w$ it follows that $-v \sim_\varepsilon -w$. Applying the previous lemma yields

$$-w < -u + \mathsf{rational}(\varepsilon).$$

Negating both sides gives

$$u - \mathsf{rational}(\varepsilon) < w.$$

□

The next result is a constructive substitute for the classical trichotomy law for linear orders. Classically, for any two reals u and v, exactly one of

$$u < v, \qquad u = v, \qquad v < u$$

holds. Constructively, however, this is too strong: strict comparison of arbitrary real numbers is not decidable in general. Weak linearity[3] captures the part of this classical intuition that remains available constructively. It asserts that whenever $u < v$, any third real w must lie on one side or the other of that strict interval, in the sense that

$$u < w \vee w < v.$$

As explained in [1, §11.2.1], taking $u := z - \mathsf{rational}(\varepsilon)$ and $v := z + \mathsf{rational}(\varepsilon)$ yields

$$z - \mathsf{rational}(\varepsilon) < w \vee w < z + \mathsf{rational}(\varepsilon),$$

and so we can view weak linearity as a form of linearity up to a small error. Besides being an

[3]There seems to be some variation in terminology here. The HoTT book refers to the property $\forall x, y, z,\ \ R(x,y) \to R(x,z) \vee R(z,y)$ as *weak linearity*, while it reserves *cotransitivity* for the variant $\forall x, y, z,\ \ R(x,y) \to R(x,z) \vee R(y,z)$ (the difference being in the order of $R(z,y)$ versus $R(y,z)$). Gilbert [3] uses the latter term for the former property, while Booij [8] treats both terms as synonyms for the first property. The standard library for Cubical Agda follows the HoTT book, so we keep that convention here.

important structural property of $<$ in its own right, it is also the key to proving Lemma 2.4.15 immediately afterwards.

Lemma 2.4.14 (Gilbert [3, Lemma 4.4])**.** *Strict inequality is weakly linear: for all* $u, v, w : \mathbb{R}$,

$$u < v \implies u < w \vee w < v.$$

Proof. It suffices to prove

$$q < r \implies \mathsf{rational}(q) < w \vee w < \mathsf{rational}(r)$$

for $q, r : \mathbb{Q}$. Indeed, if $u < v$ then by definition there merely exist some $q, r : \mathbb{Q}$ such that

$$u \leq \mathsf{rational}(q), \quad q < r, \quad \mathsf{rational}(r) \leq v.$$

Thus, if $\mathsf{rational}(q) < w$ or $w < \mathsf{rational}(r)$, then $u < w$ or $w < v$ follows by transitivity.

Let $q, r : \mathbb{Q}$ with $q < r$. We proceed by $\mathbb{R}$-induction on w.

The rational case follows immediately from the corresponding property for rational strict inequality.

For the limit case, let $x : \mathbb{Q}_+ \to \mathbb{R}$ be a Cauchy approximation and assume inductively that

$$s < t \implies \mathsf{rational}(s) < x_\varepsilon \vee x_\varepsilon < \mathsf{rational}(t).$$

for all $s, t : \mathbb{Q}$ with $s < t$ and all $\varepsilon : \mathbb{Q}_+$.

Define

$$s := \left(1 - \frac{1}{3}\right) q + \frac{1}{3} r,$$
$$t := \left(1 - \frac{2}{3}\right) q + \frac{2}{3} r,$$

so that $q < s < t < r$. Next define

$$\delta_1 := s - q,$$

$$\delta_2 := r - t,$$

$$\delta := \frac{\min(\delta_1, \delta_2)}{2}.$$

Then $0 < \delta < \delta_1, \delta_2$. Applying the inductive hypothesis to $s < t$ and x_δ yields

$$\mathsf{rational}(s) < x_\delta \vee x_\delta < \mathsf{rational}(t),$$

and hence there are two cases.

1. Suppose $\mathsf{rational}(s) < x_\delta$. Since $x_\delta \sim_{(\delta_1 - \delta) + \delta} \mathsf{limit}(x)$, Corollary 2.4.13 gives

$$\mathsf{rational}(q) = \mathsf{rational}(s - \delta_1) < \mathsf{limit}(x).$$

2. Suppose $x_\delta < \mathsf{rational}(t)$. Since $x_\delta \sim_{(\delta_2 - \delta) + \delta} \mathsf{limit}(x)$, it follows by Lemma 2.4.12 that

$$\mathsf{limit}(x) < \mathsf{rational}(t + \delta_2) = \mathsf{rational}(r).$$

In either case, we obtain

$$\mathsf{rational}(q) < \mathsf{limit}(x) \vee \mathsf{limit}(x) < \mathsf{rational}(r),$$

as required. □

Weak linearity now allows us to prove that every real is strictly smaller than any positive rational perturbation of itself, which is the final result needed before the characterization theorem.

Lemma 2.4.15 (Gilbert [3, Lemma 4.5]). *For every $u : \mathbb{R}$ and every $\varepsilon : \mathbb{Q}_+$, we have $u < u + \mathsf{rational}(\varepsilon)$.*

Proof. We proceed by $\mathbb{R}$-induction on u.

For the rational case, note that rational addition respects strict inequality, so

$$q = q + 0 < q + \varepsilon$$

and hence the desired result

$$\mathsf{rational}(q) < \mathsf{rational}(q) + \mathsf{rational}(\varepsilon)$$

follows by the strict monotonicity of the rational constructor.

For the limit case, let $x : \mathbb{Q}_+ \to \mathbb{R}$ be a Cauchy approximation and suppose inductively

$$x_\eta < x_\eta + \mathsf{rational}(\zeta)$$

for all $\eta, \zeta : \mathbb{Q}_+$.

Let $\varepsilon : \mathbb{Q}_+$. Define $\delta := \frac{\varepsilon}{5}$. By the inductive hypothesis, we have

$$x_\delta < x_\delta + \mathsf{rational}(\delta).$$

Since $x_\delta \sim_{\delta+\delta} \mathsf{limit}(x)$, Lemma 2.4.12 yields

$$\mathsf{limit}(x) < (x_\delta + \mathsf{rational}(\delta)) + \mathsf{rational}(\delta + \delta) = x_\delta + \mathsf{rational}(3\delta). \tag{2.5}$$

Then, applying the weak linearity of $<$ (Lemma 2.4.14) to (2.5), we obtain either

$$\mathsf{limit}(x) < \mathsf{limit}(x) + \mathsf{rational}(\varepsilon),$$

the desired result, or

$$\mathsf{limit}(x) + \mathsf{rational}(\varepsilon) < x_\delta + \mathsf{rational}(3\delta). \tag{2.6}$$

We show that (2.6) leads to a contradiction. Since $x_\delta \sim_{\delta+\delta} \mathsf{limit}(x)$, Lemma 2.4.11 yields

$$x_\delta \leq \mathsf{limit}(x) + \mathsf{rational}(\delta + \delta).$$

By Lemma 2.4.6, adding 3δ to both sides, we obtain

$$\begin{aligned} x_\delta + \mathsf{rational}(3\delta) &\le \mathsf{limit}(x) + \mathsf{rational}(\delta+\delta) + \mathsf{rational}(3\delta) \\ &= \mathsf{limit}(x) + \mathsf{rational}(5\delta) \\ &= \mathsf{limit}(x) + \mathsf{rational}(\varepsilon). \end{aligned}$$

Since we assumed $\mathsf{limit}(x) + \mathsf{rational}(\varepsilon) < x_\delta + \mathsf{rational}(3\delta)$, this implies

$$\mathsf{limit}(x) + \mathsf{rational}(\varepsilon) < \mathsf{limit}(x) + \mathsf{rational}(\varepsilon),$$

contradicting the irreflexivity of $<$. □

We can now assemble the preceding lemmas into the desired alternative characterization of strict inequality.

Theorem 2.4.16 (Gilbert [3, Lemma 4.1])**.** *Let $u, v : \mathbb{R}$. Then $u < v$ if and only if there merely exists some $\varepsilon : \mathbb{Q}_+$ such that $u + \mathsf{rational}(\varepsilon) \le v$.*

Proof. ($\Longrightarrow$) Suppose $u < v$. By definition, there merely exist some $q, r : \mathbb{Q}$ such that

$$u \le \mathsf{rational}(q), \quad q < r, \quad \mathsf{rational}(r) \le v.$$

Let $\varepsilon := r - q$. We have $\varepsilon > 0$ since $q < r$. By the monotonicity of addition with respect to $\le$ (Lemma 2.4.6), we have

$$u + \mathsf{rational}(\varepsilon) \le \mathsf{rational}(q) + \mathsf{rational}(\varepsilon) = \mathsf{rational}(r) \le v.$$

($\Longleftarrow$) Suppose there merely exists some $\varepsilon : \mathbb{Q}_+$ such that $u + \mathsf{rational}(\varepsilon) \leq v$. By Lemma 2.4.15, we obtain

$$u < u + \mathsf{rational}(\varepsilon) \leq v,$$

so the result follows by the transitivity of $<$ and $\leq$. □

This alternative characterization is the key result supplied by Gilbert's strategy. The original definition of $u < v$ is expressed using the existence of rationals separating u and v, but Theorem 2.4.16 shows that this is equivalent to a formulation involving only addition and the non-strict order $\leq$. Since we previously established that addition preserves and reflects $\leq$ in Lemma 2.4.6, the desired compatibility of addition with strict inequality now follows formally.

Lemma 2.4.17. *For each real $a : \mathbb{R}$, the maps*

$$u \mapsto a + u \qquad \textit{and} \qquad u \mapsto u + a$$

are monotone and order-reflecting with respect to $<$. Explicitly,

$$u < v \iff a + u < a + v$$

and hence also

$$u < v \iff u + a < v + a$$

for all $a, u, v : \mathbb{R}$.

Proof. By commutativity of addition, it suffices to prove the claim for addition on the left.

($\Longrightarrow$) Suppose $u < v$. Then by Theorem 2.4.16, there merely exists some $\varepsilon : \mathbb{Q}_+$ with $u + \mathsf{rational}(\varepsilon) \leq v$. It follows by the monotonicity of addition with respect to $\leq$ that

$$(a + u) + \mathsf{rational}(\varepsilon) \leq a + v,$$

and hence $a + u < a + v$, again by Theorem 2.4.16.

($\Longleftarrow$) Conversely, suppose $a+u < a+v$. Applying the implication just proved by adding $-a$ to both sides, we get

$$(-a) + (a + u) < (-a) + (a + v).$$

Simplifying both sides gives $u < v$. □

Before turning to the construction of multiplication and reciprocal, we record one further consequence of the theory for strict order developed above. For reciprocal, it is not enough to know merely that a real is not equal to zero; rather, we need positive information that it lies on one side of zero or the other. This is expressed by an apartness relation of $\mathbb{R}$, defined as the symmetric closure of $<$:

$$u \# v := (u < v) + (v < u).$$

Here we use the ordinary coproduct $+$, rather than propositionally truncated logical disjunction $\vee$. The coproduct itself is already a proposition, since the two injections are mutually exclusive and both propositionally-valued. Thus a witness of $x \# 0$ can be analyzed by cases, giving exactly the computational content needed to define reciprocal separately for positive and negative reals. Because $<$ is a strict order, its symmetric closure is irreflexive, symmetric, and cotransitive, so # is an apartness relation in the standard constructive sense [9].

For the third case study, we focus on the construction of multiplication and reciprocal on $\mathbb{R}$. Unlike addition, negation, min, and max, these operations cannot be obtained by a direct application of the extension principles of Section 2.3. The difficulty is that neither operation is globally Lipschitz: multiplication admits no uniform Lipschitz constant in one variable without a bound on the other, while reciprocal fails to be Lipschitz near zero. The HoTT book addresses this problem indirectly by first constructing the squaring operation $u \mapsto u^2$ and then defining multiplication using the identity

$$u \cdot v = \frac{(u+v)^2 - u^2 - v^2}{2}$$

[1]. We instead follow a more direct strategy due to Gilbert [3]. The basic idea is to define each operation first on restricted domains where the missing Lipschitz bounds become available again. For multiplication, this means fixing a bound on one factor, while for reciprocal it means restricting to reals bounded below by a positive rational. The main work is then to show that these locally defined maps agree whenever two such bounds overlap. In other words, the output of the local map must be independent of the particular chosen bound. Combined with the mere existence of suitable bounds, we can pass to the desired global maps. To accomplish this formally, Gilbert appeals to a principle of definition by surjection, which may be viewed as a consequence of the universal property of set quotients. We follow the same mathematical strategy, but adapt its implementation to the infrastructure already available in the Cubical Agda standard library. In particular, we make use of Kraus's characterization

of maps out of propositional truncations into sets [6]. We describe the construction of multiplication in some detail, and then summarize the construction of reciprocal, since it follows the same general pattern.

We begin with multiplication on the left by a fixed rational. In general, a Lipschitz constant bounds the factor by which a map can expand distances, so for multiplication by a fixed scalar $q : \mathbb{Q}$ we expect multiplication by q to be Lipschitz with constant $|q|$. We take the absolute value because distances are nonnegative, and to fit the HoTT book's convention that Lipschitz constants be positive rationals (see Definition 2.3.1), we replace it with the positive bound $\max(|q|, 1)$.

Lemma 2.4.18. *Let* $q : \mathbb{Q}$*. Then the map*

$$r \mapsto \mathsf{rational}(q \cdot r) : \mathbb{Q} \to \mathbb{R}$$

is Lipschitz with constant $\max(|q|, 1)$*.*

Proof. Suppose $r, s : \mathbb{Q}$ satisfy $|r - s| < \varepsilon$ for some $\varepsilon : \mathbb{Q}_+$. Then

$$|q \cdot r - q \cdot s| = |q| \cdot |r - s| \leq \max(|q|, 1) \cdot |r - s| < \max(|q|, 1) \cdot \varepsilon.$$

Hence

$$\mathsf{rational}(q \cdot r) \sim_{\max(|q|,1)\cdot\varepsilon} \mathsf{rational}(q \cdot s),$$

so the displayed map is Lipschitz with constant $\max(|q|, 1)$. $\square$

Applying the Lipschitz extension principle (Lemma 2.3.1) to Lemma 2.4.18, we obtain a function

$$m : \mathbb{Q} \to \mathbb{R} \to \mathbb{R}$$

where for each $q : \mathbb{Q}$ the map $m_q : \mathbb{R} \to \mathbb{R}$ extends left multiplication by q. In particular,

$$m_q(\mathsf{rational}(r)) = \mathsf{rational}(q \cdot r)$$

for all $q, r : \mathbb{Q}$. Since each m_q is obtained by Lipschitz extension, it is again Lipschitz with constant $\max(|q|, 1)$. This completes the first extension step, extending multiplication by a fixed rational from $\mathbb{Q}$ to $\mathbb{R}$. To extend in the remaining argument, we now fix a real $v : \mathbb{R}$ and ask whether the map

$$q \mapsto m_q(v)$$

is Lipschitz in the rational parameter q. This is not true uniformly in v, but it does hold once $|v|$ is bounded by a positive rational.

Lemma 2.4.19. *Let $L : \mathbb{Q}$ be positive, and let $v : \mathbb{R}$ satisfy $|v| \leq \mathsf{rational}(L)$. Then the map*

$$q \mapsto m_q(v) : \mathbb{Q} \to \mathbb{R}$$

is L-Lipschitz.

Proof. We use two properties of the family m_q, both obtained by proving the corresponding rational identities and then lifting them to the reals by uniqueness of continuous extensions.

First, for all $q, r : \mathbb{Q}$ and $u : \mathbb{R}$,

$$|m_q(u) - m_r(u)| = m_{|q-r|}(|u|).$$

Second, for every nonnegative rational s and all $u, w : \mathbb{R}$,

$$\max(m_s(u), m_s(w)) = m_s(\max(u, w)).$$

The second identity implies that m_s is monotone whenever $s \geq 0$.

Now let $q, r : \mathbb{Q}$, let $\varepsilon : \mathbb{Q}_+$, and assume

$$|q - r| < \varepsilon.$$

Then

$$|m_q(v) - m_r(v)| = m_{|q-r|}(|v|) \leq m_{|q-r|}(\mathsf{rational}(L)) = \mathsf{rational}(|q - r|L)$$

by the first property, monotonicity of $m_{|q-r|}$, the bound $|v| \leq \mathsf{rational}(L)$, and the fact that m computes on rational inputs.

By Theorem 2.4.10, it follows that the map $q \mapsto m_q(v)$ is L-Lipschitz. □

Applying Lemma 2.3.1 once more, we obtain that whenever $L : \mathbb{Q}_+$ and $v : \mathbb{R}$ satisfy $|v| \leq \mathsf{rational}(L)$, the map

$$q \mapsto m_q(v)$$

extends from $\mathbb{Q}$ to a map

$$b_{L,v} : \mathbb{R} \to \mathbb{R}$$

As a consequence of the Lipschitz extension principle, this bounded multiplication map satisfies

$$b_{L,v}(\mathsf{rational}(q)) = m_q(v)$$

for all $q : \mathbb{Q}$ and is Lipschitz with constant L. Thus, once a rational bound on $|v|$ has been chosen, right multiplication by v is defined on all real inputs.

This completes the local part of the construction of multiplication. To pass from the bound-indexed local maps $b_{L,v}$ to genuine multiplication defined on all of $\mathbb{R}$, two requirements must be met. First, suitable rational bounds must merely exist for every real. Second, whenever two such bounds are both valid, the corresponding bounded multiplication maps must agree. The next lemma supplies the first of these requirements.

Lemma 2.4.20 (Gilbert [3, Lemma 4.13])**.** *For every $u : \mathbb{R}$, there merely exists a $q : \mathbb{Q}_+$ such that $|u| < \mathsf{rational}(q)$.*

Proof. Applying Lemma 2.4.15 to $|u|$ with $\varepsilon := 1$, we obtain

$$|u| < |u| + 1.$$

By the Archimedean property (Theorem 2.4.7), there merely exists some $q : \mathbb{Q}$ such that

$$|u| < \mathsf{rational}(q) < |u| + 1.$$

Since $|u| \geq 0$, it follows that $0 < \mathsf{rational}(q)$. Therefore there merely exists $q : \mathbb{Q}_+$ such that

$$|u| < \mathsf{rational}(q).$$

□

Our remaining goal is to fulfill the second requirement by eliminating the dependence on the chosen bound. We briefly compare two ways of making this precise: Gilbert's principle of definition by surjection and the characterization of maps of the form $\|A\| \to B$ where B is a set due to Kraus, which is what we use in the formalization.

Following Kraus, a map $f : A \to B$ is **weakly constant** if it comes equipped with an element of type

$$\mathsf{IsWeaklyConstant}(f) := \prod\nolimits_{(x,y:A)} f(x) = f(y).$$

In general, weak constancy is not enough to define a map out of a proposition truncation, since the paths

$$f(x) = f(y)$$

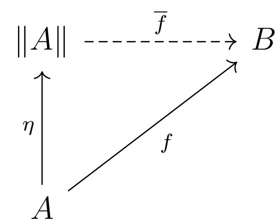
may themselves need to satisfy higher coherence conditions. When the codomain B is a set, however, all such higher conditions are automatic, so weak constancy is already sufficient.

Theorem 2.4.21 (Kraus [6, Proposition 2.2])**.** *If A is a type and B is a set, there is an equivalence*

$$(\|A\| \to B) \quad \simeq \quad \sum\nolimits_{(f:A\to B)} \mathsf{IsWeaklyConstant}(f).$$

The consequence of Theorem 2.4.21 that we use here is that every weakly constant map

$f : A \to B$ into a set B extends uniquely to a map $\overline{f} : \|A\| \to B$. This is summarized by the commutative diagram



where $\eta : A \to \|A\|$ is the canonical map into the propositional truncation. For this approach, the relevant data consists of a map $f : A \to B$, the required invariance is weak constancy, and the output is the extension $\overline{f} : \|A\| \to B$.

Gilbert packages the same basic idea in a different way, using a principle of **definition by surjection** [3, Definition 4.9]. Suppose $f : A \to C$ and $g : A \to B$ are maps such that g is surjective and f respects the equivalence relation on A induced by g, defined by

$$a_1 \sim_g a_2 := g(a_1) = g(a_2).$$

Under these hypotheses, the universal property of the set quotient by $\sim_g$ yields a map

$$\overline{f} : A/{\sim_g} \to C$$

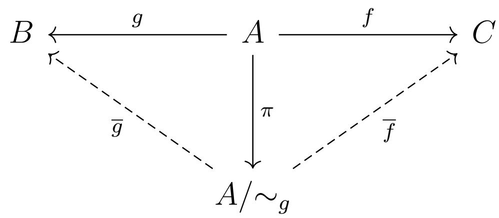
extending f. The same universal property also gives a map

$$\overline{g} : A/{\sim_g} \to B$$

extending g, and the surjectivity of g implies that $\overline{g}$ is an equivalence. We may therefore define

$$h := \overline{f} \circ \overline{g}^{-1} : B \to C,$$

which satisfies $h \circ g = f$. This setup is represented by the commutative diagram



where $\pi : A \to A/{\sim_g}$ is the canonical projection onto the quotient. Here the relevant data are the maps $f : A \to C$ and $g : A \to B$, the required invariance is that f respects the equivalence relation induced by g, and the output is the induced map $h : B \to C$.

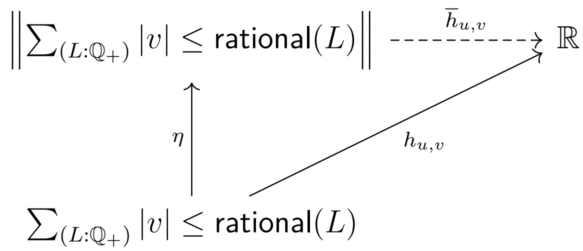
We now show how both abstract principles can be instantiated in the construction of multiplication, thereby eliminating the dependence on the chosen bound. To separate the real inputs from the chosen bound on the second argument, for each $u, v : \mathbb{R}$, we define a map

$$\begin{aligned}
&h_{u,v} : \left(\sum\nolimits_{(L:\mathbb{Q}_+)} |v| \leq \mathsf{rational}(L) \right) \to \mathbb{R} \\
&h_{u,v}(L, \varphi, \psi) := b_{L,v}(u).
\end{aligned}$$

Since $\mathbb{R}$ is a set by Corollary 2.2.2, Kraus's theorem applies directly to this situation. Once $h_{u,v}$ is shown to be weakly constant, it extends uniquely to a map

$$\overline{h}_{u,v} : \left\| \sum\nolimits_{(L:\mathbb{Q}_+)} |v| \leq \mathsf{rational}(L) \right\| \to \mathbb{R}$$

as summarized by the commutative diagram:



By Lemma 2.4.20, the propositional truncation

$$\left\| \sum_{(L:\mathbb{Q}_+)} |v| \leq \mathsf{rational}(L) \right\|$$

is always inhabited. Evaluating $\overline{h}_{u,v}$ at the truncated witness therefore yields multiplication.

We can achieve the same result using Gilbert's approach. Fix $u : \mathbb{R}$ and define

$$A_u := \sum_{(v:\mathbb{R})} \sum_{(L:\mathbb{Q}_+)} |v| \leq \mathsf{rational}(L),$$

$$f_u : A_u \to \mathbb{R},$$

$$f_u(v, L, \varphi, \psi) := b_{L,v}(u),$$

$$g_u : A_u \to \mathbb{R},$$

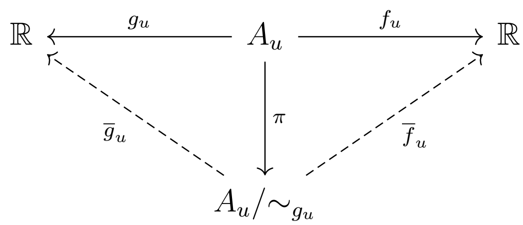
$$g_u(v, L, \varphi, \psi) := v.$$

By Lemma 2.4.20, the map g_u is surjective. Since g_u forgets the chosen bound, proving that f_u respects $\sim_{g_u}$ amounts to showing that f_u is invariant with respect to the choice of bound.

Once this is known, we obtain a map

$$\mu_u : \mathbb{R} \to \mathbb{R}$$

as summarized by the commutative diagram



Multiplication is then given by

$$u \cdot v := \mu_u(v).$$

Both principles therefore depend exactly on the two requirements stated above: first, that every real is merely bounded by a positive rational, and second, that bounded multiplication is invariant with respect to the chosen bound. Since Kraus's formulation is more direct and is already available in the Cubical Agda library, it is the one we adopt here. It therefore remains to establish the required invariance.

Lemma 2.4.22. *For every $u, v : \mathbb{R}$, the map*

$$h_{u,v} : \left(\sum_{(L:\mathbb{Q}_+)} |v| \le \mathsf{rational}(L) \right) \to \mathbb{R}$$

is weakly constant. Explicitly, if $(L, \varphi, \psi), (M, \omega, \chi) : \sum_{(L:\mathbb{Q}_+)} |v| \le \mathsf{rational}(L)$ then

$$h_{u,v}(L, \varphi, \psi) = b_{L,v}(u) = b_{M,v}(u) = h_{u,v}(M, \omega, \chi).$$

Proof. Both $b_{L,v}$ and $b_{M,v}$ are continuous maps $\mathbb{R} \to \mathbb{R}$, since each is obtained by Lipschitz extension. Moreover, they agree on rational inputs: for every $q : \mathbb{Q}$,

$$b_{L,v}(\mathsf{rational}(q)) = m_q(v) = b_{M,v}(\mathsf{rational}(q)).$$

By Lemma 2.3.3, it follows that

$$b_{L,v} = b_{M,v}$$

as functions $\mathbb{R} \to \mathbb{R}$. Evaluating both sides at u yields

$$b_{L,v}(u) = b_{M,v}(u),$$

so $h_{u,v}$ is weakly constant. □

Combining Lemma 2.4.20 with the previous lemma, and using that $\mathbb{R}$ is a set, applying Kraus's theorem to the map $h_{u,v}$ yields a global multiplication operation

$$\cdot : \mathbb{R} \to \mathbb{R} \to \mathbb{R}.$$

For any $L : \mathbb{Q}_+$ and $u, v : \mathbb{R}$ such that $|v| \leq \mathsf{rational}(L)$, multiplication satisfies

$$u \cdot v = b_{L,v}(u).$$

In particular, multiplication computes on rational inputs as expected: for all $q, r : \mathbb{Q}$,

$$\mathsf{rational}(q) \cdot \mathsf{rational}(r) = \mathsf{rational}(q \cdot r).$$

The agreement with the bounded multiplication map also transfers the Lipschitz estimate established above for the map $u \mapsto b_{L,v}(u)$.

Lemma 2.4.23 (Gilbert [3, Lemma 4.18])**.** *Let $L : \mathbb{Q}_+$ and let $v : \mathbb{R}$ satisfy*

$$|v| \leq \mathsf{rational}(L).$$

Then the map

$$u \mapsto u \cdot v$$

is Lipschitz with constant L.

Proof. By construction, the map $u \mapsto u \cdot v$ agrees with $b_{L,v}$. Since $b_{L,v}$ was obtained by Lipschitz extension of the L-Lipschitz map $q \mapsto m_q(v)$, it is itself L-Lipschitz. Therefore $u \mapsto u \cdot v$ is L-Lipschitz as well. □

We next show that multiplication is continuous in each argument separately. Continuity in the first argument is immediate from the previous lemma.

Lemma 2.4.24 (Gilbert [3, Lemma 4.19])**.** *For all $v : \mathbb{R}$, the map*

$$u \mapsto u \cdot v : \mathbb{R} \to \mathbb{R}$$

is continuous.

Proof. By Lemma 2.4.20, there merely exists a positive rational L such that

$$|v| \leq \mathsf{rational}(L).$$

For any such L, Lemma 2.4.23 shows that the map $u \mapsto u \cdot v$ is L-Lipschitz, and therefore continuous. □

Continuity in the second argument is less direct. The construction of multiplication proceeded by fixing a bound on the second factor and extending in the first, so a separate argument is needed for continuity in the second variable.

Lemma 2.4.25. *For every $u : \mathbb{R}$, the map*

$$v \mapsto u \cdot v : \mathbb{R} \to \mathbb{R}$$

is continuous.

Proof. Our argument rests on the identity

$$|a \cdot u - a \cdot v| = |a| \cdot |u - v|,$$

which holds for all $a, x, y : \mathbb{R}$ and is obtained by lifting the corresponding rational formula using the uniqueness of continuous extensions. Note that, in verifying the continuity hypotheses needed for that lifting step, we only use the continuity of multiplication in the first argument.

Fix $v : \mathbb{R}$ and $\varepsilon : \mathbb{Q}_+$. By Lemma 2.4.20, there merely exists a positive rational η such that $|u| \leq \mathsf{rational}(\eta)$. Choose $\delta := \frac{\varepsilon}{2\eta}$. If $v \sim_\delta w$ then Theorem 2.4.10 gives

$$|v - w| < \mathsf{rational}(\delta).$$

Hence

$$|u \cdot v - u \cdot w| = |u| \cdot |v - w| \leq \mathsf{rational}(\eta) \cdot \mathsf{rational}(\delta) = \mathsf{rational}(\varepsilon/2) < \mathsf{rational}(\varepsilon).$$

Applying Theorem 2.4.10 again, we conclude $u \cdot v \sim_\varepsilon u \cdot w$. Therefore $v \mapsto u \cdot v$ is continuous.

□

Because multiplication agrees with rational multiplication on rational inputs and is continuous in each variable separately, commutativity, associativity, the unit laws, and distributivity over addition lift from $\mathbb{Q}$ to $\mathbb{R}$. Hence $\mathbb{R}$ carries the structure of a commutative ring.

Reciprocal is constructed by the same local-to-global strategy, so we only sketch the main ideas. The difference is that the relevant local domains are no longer determined by upper bounds on magnitude, but by positive lower bounds away from zero. Fix $\delta : \mathbb{Q}_+$ and consider the rational map

$$q \mapsto \frac{1}{\max(q, \delta)}.$$

Because the denominator is bounded below by δ, this map is Lipschitz with constant $\frac{1}{\delta^2}$, and therefore extends to a map

$$r_\delta : \mathbb{R} \to \mathbb{R}.$$

Intuitively, r_δ is the reciprocal function clamped to the interval $[\delta, \infty)$.

As with multiplication, the key step is to verify that these bounded maps agree whenever two lower bounds are both valid. Specifically, if δ_1 and δ_2 are two positive rational lower bounds that are both valid for the same real u, then the corresponding local maps agree at u. Since every positive real is merely bounded below by some positive rational, the same truncation argument used for multiplication yields a globally defined reciprocal on the positive reals.

Finally, the apartness relation introduced at the end of the previous case study allows us to extend this positive reciprocal to all reals apart from zero. A witness of $u \# 0$ provides a case split into $0 < u$ or $u < 0$. In the positive case we use the construction above; in the negative case we reduce to the positive reciprocal of $-u$ and then negate. The resulting operation can be shown to satisfy the property that every real is apart from zero if and only if it is invertible.

We can now assemble the algebraic and order-theoretic structure developed in the three case studies into the constructive definition of an Archimedean ordered field used in the HoTT book.

Definition 2.4.1 (The Univalent Foundations Program [1, Definition 11.2.7])**.** An **ordered field** consists of a set F together with constants 0, 1, operations $-$, $+$, $\cdot$, min, max, and mere relations $\leq$, $<$, $\#$ such that:

- $(F, 0, 1, +, -, \cdot)$ is a commutative ring.
- $(F, \leq, \min, \max)$ is a lattice.
- The relation $<$ is a strict order, meaning it is irreflexive, transitive, and weakly linear.

- The relation $\#$ is an apartness relation, meaning it is irreflexive, symmetric, and cotransitive.
- Every element $x : F$ is invertible if and only if $x \# 0$.
- For every $x, y, z : F$:

$$
\begin{aligned}
x \leq y &\iff \neg(y < x), & x < y \leq z &\implies x < z,\\
x \# y &\iff (x < y) + (y < x), & x \leq y < z &\implies x < z,\\
x \leq y &\iff x + z \leq y + z, & (x \leq y) \times (0 \leq z) &\implies xz \leq yz,\\
x < y &\iff x + z < y + z, & 0 < z &\implies (x < y \iff xz < yz),\\
0 < x + y &\implies 0 < x \vee 0 < y, & 0 &< 1.
\end{aligned}
$$

Every ordered field comes equipped with a canonical embedding $r : \mathbb{Q} \to F$. An ordered field is **Archimedean** if for every $x, y : F$ with $x < y$ there exists a $q : \mathbb{Q}$ such that $x < r(q) < y$.

Theorem 2.4.26. *The HoTT book reals form an Archimedean ordered field.*

The preceding case studies establish the main requirements of this theorem. The operations of negation, addition, multiplication, together with their algebraic laws, give the set $\mathbb{R}$ the structure of a commutative ring, while min and max induce the non-strict order relation $\leq$ and satisfy the lattice laws. The relation $<$ was shown to be a strict order satisfying the Archimedean property, and both $\leq$ and $<$ were shown to be preserved and reflected by addition. Apartness was defined as the symmetric closure of $<$, and reciprocal was constructed on exactly those reals apart from zero. The remaining compatibility laws not discussed explicitly are verified in the Agda formalization and are established using the same main proof patterns illustrated in this chapter.

3 - REAL NUMBERS IN CUBICAL AGDA

This chapter describes the Cubical Agda implementation of the mathematics from Chapter 2. The implementation is the key technical contribution of this thesis: it formalizes all definitions, theorems, and lemmas from the previous chapter, as well as their proofs, as machine-checked code (over 13,000 lines). The code is open source and available at `https://github.com/utahplt/hott-reals`.

The chapter is organized as follows. Section 3.1 gives a brief example showing that arithmetic on the reals computes definitionally when applied to rational inputs. Section 3.2 surveys the structure of the codebase, following the organization of Chapter 2. Section 3.3 describes our use of Claude Code during the development. Section 3.4 reflects on three insights that emerged during the formalization. We do not walk through Agda proofs in detail; the repository is the source of truth for the code itself.

3.1 - A COMPUTATIONAL EXAMPLE

Because the operations on the HoTT book reals are defined by recursion on the higher inductive type, and the rational constructor computes definitionally, arithmetic identities between rational real numbers hold by definition. For example, the identity $\mathsf{rational}(2) + \mathsf{rational}(2) = \mathsf{rational}(4)$ is proved by `refl`. The type checker verifies this directly.

```
2+2≡4 : (rational 2) + (rational 2) ≡ rational 4
2+2≡4 = refl
```

In Lean 4 (v4.29.0) with Mathlib (v4.29.0), the analogous statement does not hold definitionally.

```
theorem two_add_two_equal_four_real : (2 : ℝ) + (2 : ℝ) = (4 : ℝ) := rfl
```

```
-- Not a definitional equality: the left-hand side
--   2 + 2
-- is not definitionally equal to the right-hand side
--   4
```

This is not optimal. The construction of the reals in Lean uses a quotient of Cauchy sequences in a way that does not support direct normalization.

3.2 - ORGANIZATION

The Cubical Agda implementation comprises 33 modules and approximately 13,500 lines of code. Its organization parallels that of Chapter 2, together with supporting infrastructure for the rationals that was not available in the Cubical Agda standard library. Table 3.1 summarizes the distribution of code by topic. In this section we describe our use of the Cubical Agda standard library and the rational support infrastructure, and then survey the code corresponding to each section of Chapter 2.

Topic	Main modules	Lines
Rational support infrastructure	`Data.Rationals.Order`, `Data.Rationals.Properties`	2,293
Definition, induction, and recursion	`Data.Real.Base`, `Data.Real.Definitions`, `Data.Real.Induction`	679
Closeness	`Data.Real.Close.ReflexiveSymmetric`, `Data.Real.Close.CloseAlternative`, `Data.Real.Close.Other`	2,484
Continuity and extension	`Data.Real.Lipschitz.Base`, `Data.Real.Lipschitz.Closed`, `Data.Real.Nonexpanding`, `Data.Real.Properties`	1,887
Algebra	`Data.Real.Algebra.*`	2,922
Order	`Data.Real.Order.*`	3,295
Total		13,560

Table 3.1: Distribution of the Agda codebase by topic.

The formalization is built on the Cubical Agda standard library, which supplies the foundations for doing formalized mathematics in Cubical Type Theory [10]. We make extensive use of its propositional truncation module, including eliminators at various arities and the `SetElim` module, which provides the Cubical library's version of Kraus's characterization of maps from propositional truncations into sets [6]. This is used in the construction of both multiplication and reciprocal (see Section 2.4). The library also provides the theorem that any type equipped with a reflexive mere relation implying identity is a set [1, Theorem 7.2.2], which we use to show that $\mathbb{R}$ is a set. For order theory, we rely on the library's vocabulary of posets, strict orders, and apartness relations, including the result that the symmetric closure of a strict order is an apartness relation. On the algebraic side, the library supplies definitions for groups, rings, and fields, which we use to package the algebraic structure of $\mathbb{Q}$ and $\mathbb{R}$. The library's notion of field is that of a denial field in the sense of Mines et al. [9], requiring inverses for elements satisfying $\neg(x = 0)$ rather than elements apart from zero. Since the ordered field definition of the HoTT book (Definition 2.4.1) uses an apartness relation, we were unable to use the library's field definition for $\mathbb{R}$. We did, however, use the standard definition for $\mathbb{Q}$.

At the start of the project, the standard library did not include an instance showing that $\mathbb{Q}$ is a denial field, nor did it provide an explicit reciprocal operator for the rationals. We contributed a rational denial field instance as a pull request (`https://github.com/agda/cubical/pull/1260`) to the Cubical Agda standard library, which was later accepted.

The supporting infrastructure for the rationals totals approximately 2,300 lines across two modules. The Cubical Agda standard library provides the basic definitions and algebraic operations for the rationals, but at the time of writing there are many missing definitions and results that our formalization requires. We therefore develop the missing results ourselves:

- We formulate missing interactions between the algebraic and order-theoretic structure, namely the monotonicity and order-reflection of addition and multiplication, the fact that negation is antitone, and the fact that reciprocal on positive rationals is antitone, all with respect to both $\leq$ and $<$.

- We build the lattice structure on $\mathbb{Q}$, showing that max and min form join- and meet-semilattices respectively, and establish the midpoint identities for max and min mentioned in Lemma 2.4.1.

- We define the absolute value and distance functions on the rationals and develop their basic theory, including the triangle inequality and reverse triangle inequality in both norm and distance forms, the metric axioms for distance, and the norm axioms for absolute value.

With these contributions in place, we prove that the rational versions of each operation defined on the reals are Lipschitz or non-expanding. We also develop lemmas about the closeness relation on the rationals, including the openness and separatedness properties. Other supporting results include strict monotonicity and bounds for affine combinations, clamping operations with interval membership, and midpoint inequalities such as $q < \frac{q+r}{2} < r$. Finally, we prove the rational identities that are subsequently lifted to $\mathbb{R}$ to establish the Archimedean ordered field axioms.

The code corresponding to Section 2.1 lives in three modules. The `Base` module defines the higher inductive-inductive type $\mathbb{R}$ and the closeness relation. The code follows Definition 2.1.1 directly, except that positivity witnesses such as $0 < \varepsilon$ and Cauchy approximation witnesses are carried explicitly throughout, whereas the informal presentation of Chapter 2 routinely suppresses them. Cubical Agda's native support for higher inductive types allows the mutually dependent definitions of $\mathbb{R}$ and $\sim$, including the path constructor, to be expressed directly; Figure 3.1 shows the definition as it appears in the code.

In addition to general $(\mathbb{R}, \sim)$-induction and its specializations $\mathbb{R}$-induction and $\sim$-induction of Definitions 2.1.2 to 2.1.4, the `Induction` module also formulates proposition-valued variants that discharge the path hypotheses automatically, as well as a binary proposition-valued variant for motives indexed by two reals. The enhanced $(\mathbb{R}, \sim)$-recursion principle of Definition 2.1.5 is also defined here. Because Cubical Agda has native support for higher inductive types, the computation rules corresponding to all of these principles hold definitionally. The `Definitions` module collects auxiliary definitions used by the first three sections, including the dependent Cauchy approximation conditions of Section 2.1, the Lipschitz, non-expanding, and continuity conditions of Section 2.3 and the triangle inequality and roundedness conditions of Section 2.2.

The code for results about closeness, corresponding to Section 2.2, is split across three submodules. The first, `ReflexiveSymmetric`, establishes reflexivity and symmetry of the closeness relation (Lemmas 2.2.1 and 2.2.3). The second, `CloseAlternative`, contains the construction of the alternative closeness relation $\approx$ of Theorem 2.2.5. At nearly 1,850 lines, this is the single largest proof in the formalization, despite receiving a relatively compact treatment in Section 2.2. The three submodules are separated primarily because iteratively type-checking the alternative closeness proof became a bottleneck during development; splitting it into its own module kept the feedback loop manageable. The third submodule, `Other`, proves the equivalence of the alternative closeness relation with the inductively defined closeness relation and records the resulting roundedness and triangle inequality properties. The

```
data ℝ : Type

data Close : (ε : ℚ) → (0 < ε) → ℝ → ℝ → Type ℓ-zero

syntax Close ε p x y = x ~[ ε , p ] y

CauchyApproximation : ((ε : ℚ) → 0 < ε → ℝ) → Type ℓ-zero
CauchyApproximation x =
  ((δ ε : ℚ) (p : 0 < δ) (q : 0 < ε) →
   x δ p ~[ δ + ε , 0<+' {x = δ} {y = ε} p q ] x ε q)

data ℝ where
  rational : ℚ → ℝ
  limit : (x : (ε : ℚ) → 0 < ε → ℝ) →
          CauchyApproximation x →
          ℝ
  path : (x y : ℝ) →
         ((ε : ℚ) (p : 0 < ε) → x ~[ ε , p ] y) →
         x ≡ y

data Close where
  rationalRational :
    (q r ε : ℚ) (φ : 0 < ε) →
    - ε < q - r → q - r < ε →
    rational q ~[ ε , φ ] rational r
  rationalLimit :
    (q ε δ : ℚ) (φ : 0 < ε) (ψ : 0 < δ) (θ : 0 < ε - δ)
    (y : (ε : ℚ) → 0 < ε → ℝ) (ω : CauchyApproximation y) →
    rational q ~[ ε - δ , θ ] (y δ ψ) →
    rational q ~[ ε , φ ] (limit y ω)
  limitRational :
    (x : (ε : ℚ) → 0 < ε → ℝ) (φ : CauchyApproximation x)
    (r ε δ : ℚ) (ψ : 0 < ε) (θ : 0 < δ) (ω : 0 < ε - δ) →
    (x δ θ) ~[ ε - δ , ω ] rational r →
    limit x φ ~[ ε , ψ ] rational r
  limitLimit :
    (x y : (ε : ℚ) → 0 < ε → ℝ)
    (φ : CauchyApproximation x) (ψ : CauchyApproximation y)
    (ε δ η : ℚ) (θ : 0 < ε) (ω : 0 < δ) (π : 0 < η)
    (ρ : 0 < ε - (δ + η)) →
    (x δ ω) ~[ ε - (δ + η) , ρ ] (y η π) →
    limit x φ ~[ ε , θ ] limit y ψ
  squash :
    (ε : ℚ) (φ : 0 < ε) (u v : ℝ) →
    isProp $ u ~[ ε , φ ] v
```

Figure 3.1: The higher inductive-inductive definition of the HoTT book reals and the closeness relation in Cubical Agda, corresponding to Definition 2.1.1. The forward declarations of ℝ and `Close` establish the mutual dependence before the constructors are given.

`Other` module also includes helpers that package the interaction between closeness and limits. For example, `closeLimit` derives $u \sim_{\varepsilon+\delta} \mathsf{limit}(y)$ from $u \sim_{\varepsilon} y_\delta$, supplying the necessary hypotheses for the relevant closeness constructor in both cases of an induction argument. These helpers are used throughout the algebra and order development whenever a proof needs to argue that an arbitrary real is close to the limit of a Cauchy approximation.

The code for Section 2.3 spans four modules. The `Lipschitz.Base` module proves unary Lipschitz extension (Lemma 2.3.1) using the enhanced $(\mathbb{R}, \sim)$-recursion principle, while the `Nonexpanding` module proves binary non-expanding extension (Lemma 2.3.2). The `Lipschitz.Closed` module formalizes Exercise 11.8 of the HoTT book [1], which extends the Lipschitz extension principle to maps defined on closed intervals of the rationals. We did not end up using this result, since we decided to follow Gilbert's approach for multiplication and reciprocal rather than continuing with the squaring-based construction given in the HoTT book, which makes use of this exercise. However, it may still be of independent interest for future work. The `Properties` module contains the uniqueness of continuous extensions (Lemma 2.3.3) and its binary and ternary analogues (Lemma 2.3.4), the continuous extension law helpers at arities one through three (Corollary 2.3.5), and the binary Lipschitz composition lemma (Lemma 2.4.4). These results are the main tools used to transfer algebraic identities from $\mathbb{Q}$ to $\mathbb{R}$ in Section 2.4. The `Properties` module also contains the proof that $\mathbb{R}$ is a set (Corollary 2.2.2), the surjectivity of the limit constructor, and the injectivity of the $\mathsf{rational}$ constructor, which logically belong to earlier sections but are placed here due to module dependency constraints.

The code for Section 2.4 is the largest portion of the formalization, spanning approximately 6,200 lines across thirteen modules split between algebra and order. On the algebra side, the `Negation` module lifts rational negation via Lipschitz extension, and the `Addition` module lifts rational addition via non-expanding extension. Their algebraic laws are transferred from $\mathbb{Q}$ using the continuous extension laws, and the resulting structure is packaged as an Abelian group. The `Lattice` module similarly lifts min and max via non-expanding extensions and establishes their lattice structure. The `Multiplication` module follows the iterated Lipschitz extension strategy described in Section 2.4, and the needed algebraic identities are lifted to yield a commutative ring instance. The `Reciprocal` module follows the same local-to-global pattern to obtain the reciprocal on positive reals, and then extends to all reals apart from zero by case splitting on the apartness witness. On the order side, the `Base` module defines non-strict and strict inequality as in Section 2.4. The `Magnitude` and `Distance` modules define absolute value and the distance metric on $\mathbb{R}$ and and develop their basic theory. The remaining order modules establish interactions between order and algebraic operations: monotonicity and order-reflection of addition with respect to both $\leq$ and $<$, weak linearity, Gilbert's alternative characterization of strict inequality (Theorem 2.4.16), and positivity preservation and strict monotonicity for multiplication by positive reals.

3.3 - ON THE USE OF CLAUDE CODE

We experimented with Claude Code, Anthropic's coding agent [11]. We used Claude Code to aid navigation of the Cubical Agda standard library, and as a tool for converting detailed proof sketches into formal Agda proofs.

The Cubical Agda standard library is large, and locating the right lemma or definition by name alone can be difficult. Because Claude Code has tool access to the file system, it could search the library by description rather than by name, explain unfamiliar definitions, and help navigate module structure. This made it an effective form of semantic search during

```
Can you fill in the type hole for the χ subterm in
`maxMultiplyBoundedReciprocalPositiveContinuous` in Reciprocal.agda based on the
following proof sketch?

- By `||≤rational` (Properties2.agda), there exists an L : Q
  with |max(x, δ)| ≤ L
- By the continuity of `boundedReciprocalPositive`, there is
  a θ so that x ~[θ] y ⇒
  brp(δ, _, x) ~[(L+1)⁻¹ ε/2] brp(δ, _, y)
- Choose η₂ := min(θ, 1)
- Fix y : R and assume x ~[η₂] y
- Then x ~[1] y. By `maxNonexpandingℝ₂`, we have
  max(x, δ) ~[1] max(y, δ). Applying `close⇒≤+ε` to
  |x| ≤ L we obtain |max(y, δ)| ≤ L + 1
- By `·distanceₗ`,
  |max(y, δ) · brp(δ, x) - max(y, δ) · brp(δ, y)| = |max(y, δ)| · |brp(δ, x) - brp(δ, y)|
- Then
  |max(y, δ)| · |brp(δ, x) - brp(δ, y)| ≤ (L + 1) · (L + 1)⁻¹ · ε/2 = ε/2
- Then using `close⇒distance<`, we obtain the required
  closeness result

Hint: For the very last step, you'll probably need to use
the pattern for substituting the propositional positivity
argument to `Close` used in the π term below.
```

Figure 3.2: A prompt given to Claude Code during the development of the reciprocal construction. The sketch specifies the proof strategy step by step, naming intermediate results and the library lemmas to apply. We abbreviate `boundedReciprocalPositive` as `brp` to fit the page.

development.

More surprisingly, Claude Code could take informal proof sketches and produce formal Agda proof terms that type-checked, requiring only minor cleanup afterwards. Figure 3.2 shows a representative prompt from the development. The proof sketch specifies the proof strategy step by step, naming the intermediate results and the specific library lemmas to invoke at each stage. The sketch also includes a hint about a substitution pattern for rewriting a dependent positivity witness in the final step. Given this level of detail, Claude Code produced a proof term that, after small adjustments for code style, is essentially the one that remains in the codebase as `maxMultiplyBoundedReciprocalPositiveContinuous` in `Reciprocal.agda`. The proof strategy had to be specified in the prompt, but the agent handled the translation to Agda syntax, including the selection of correct library functions, the bookkeeping of positivity witnesses, and the assembly of the proof term.

3.4 - LESSONS LEARNED

The effort of formalization surfaced three insights about the interaction between the informal presentation of the HoTT book reals and their realization in Cubical Agda. In each case, the precision required by formalization surfaced something that informal reading alone did not. We describe them in turn and then identify the patterns they share.

The first concerns the alternative closeness relation of Theorem 2.2.5. The constructors of $\sim$ provide sufficient conditions for each combination of a rational and a limit, but not necessary conditions. Given $\mathsf{limit}(x) \sim_\varepsilon \mathsf{rational}(r)$ as a hypothesis, for instance, nothing about the inductive-inductive definition lets us immediately extract a $\delta : \mathbb{Q}_+$ satisfying the condition of the limit-rational constructor. Indeed, as the HoTT book notes, inductive type families "have a tendency to contain 'more than was put into them' " [1, § 11.3.2]. The book flags this explicitly, resolving it by defining a relation $\approx$ by recursion which computes on construtors and proving that $\approx$ and $\sim$ coincide. This resolution was already in place before the formalization began. What the effort of formalization added was a much slower understanding of *why* this construction is the right response. The `CloseAlternative` module contains the longest single proof in the codebase, and only after spending time on it and then applying the alternative closeness relation in later proofs did its role become clear. In principle, each extraction of information from a closeness hypothesis could be carried out locally by an induction argument; indeed, that is exactly how the forward implication

$$u \sim_\varepsilon v \implies u \approx_\varepsilon v$$

of Theorem 2.2.6 is proved. The alternative closeness relation packages the necessary conditions once, as a globally applicable statement, so that downstream proofs can invoke a single reusable statement rather than reinvoke induction locally each time. This role was not immediately evident from reading the book alone; it became evident only after formalizing the equivalence and applying the relation in subsequent proofs.

The second insight concerns the enhanced $(\mathbb{R}, \sim)$-recursion principle of Definition 2.1.5. The HoTT book states the limit case hypothesis in a notation that writes $f(x_\varepsilon)$ in the inductive hypothesis, where f is the function being defined by recursion [1, § 11.3.2]. Both the approximation x itself and the inductively defined values $f(x_\varepsilon)$ are therefore tacitly available. Our initial, naive transcription into Agda, however, dropped x from the inductive hypothesis entirely. In the induction principle, the motive $A : \mathbb{R} \to \mathcal{U}$ is indexed by $\mathbb{R}$, so x_ε appears naturally in the type $A(x_\varepsilon)$ of the inductively assigned values. In the non-dependent recursion principle, the motive is a plain type $A : \mathcal{U}$, and no mention of x_ε remains. It is natural to drop x from the inductive hypothesis altogether, leaving only $f : \mathbb{Q}_+ \to A$. This suffices for most uses of the principle, because a typical recursion is concerned with producing output in the codomain A and has no occasion to refer back to the domain approximation. The weakness only surfaced in the inner recursion used to define alternative closeness, where the codomain is itself a family of relations on $\mathbb{R}$. In the limit-limit case there, the body of the required existential must name a real drawn from the domain approximation.

Without access to x in the recursion hypothesis, the case cannot even be formulated. We wrote more than two thousand lines of code using the weaker recursion principle before reaching this point. Repairing it required reformulating the recursion principle to carry both the domain approximation and its codomain assignment (see the comment at lines 310–325 of `Induction.agda`) and refactoring each existing call site, including the Lipschitz extension in `Lipschitz.Base`. Here the formalization did not merely deepen understanding of an existing construction; it required us to notice that the naive transcription had silently dropped information that the informal presentation carried for free.

The third insight concerns the extension of continuous functions of several variables. The HoTT book states the uniqueness of continuous extensions for maps of one variable (Lemma 2.3.3) and remarks in passing that identities in several variables can be extended in the same manner. The precise hypotheses are left unstated. In particular, it is not obvious upon first reading whether joint continuity is required or whether continuity in each variable separately suffices. To transfer algebraic identities from $\mathbb{Q}$ to $\mathbb{R}$ throughout Section 2.4, we needed to commit to one or the other. We showed in Lemma 2.3.4 that for binary functions, separate continuity in each variable is sufficient, with the proof proceeding by applying the one-variable lemma one coordinate at a time. We formulate a ternary analogue similarly in the code. Here the formalization genuinely supplied content that the informal presentation only gestures at. The informal text does not say the wrong thing, but it does not say enough to be applied directly, and making the result applicable in the formalization required articulating hypotheses and proving a new lemma.

These three instances illustrate two recurring patterns in the relationship between informal mathematics and its formalization. In the first case, the informal presentation is complete and rigorous, but formalizing it made its structure evident in a way reading alone did not. In the other two, the formal system required explicit articulation of information that the informal presentation left implicit, and supplying that information became part of the mathematical work of the formalization.

4 - RELATED WORK

The present chapter discusses the prior and concurrent work most directly related to this thesis, including other formalizations of the real numbers and broader programs in constructive analysis within univalent foundations.

The closest prior work is Gilbert's 2017 formalization of the HoTT book reals in the Rocq proof assistant [3]. Gilbert, like the HoTT book, follows O'Connor [4] in defining closeness as a family of binary relations indexed by positive rationals, rather than as a real-valued distance function. Beyond the formalization itself, Gilbert generalizes the HoTT book construction to arbitrary premetric spaces, showing that Cauchy completion is a monad on the category of premetric spaces with Lipschitz functions, and uses Altenkirch, Danielsson, and Kraus's partiality monad [12] to give a semi-decision procedure comparing a real to a rational. Rocq lacks native support for both inductive-inductive types and higher inductive types; Sozeau's experimental branch added support for inductive-inductive types, but higher inductive types remain unsupported, requiring Gilbert to axiomatize the path constructors of the HoTT book reals. This motivates our choice of Cubical Agda, which natively supports both. We adopt Gilbert's local-to-global strategy for multiplication and reciprocal and his alternative characterization of strict inequality, and depart in two respects. Cubical Agda supplies the path constructors and their computation rules definitionally, and we use Kraus's [6] theorem on maps out of propositional truncations into sets in place of Gilbert's definition by surjection to package the local-to-global step.

Concurrently, Molena et al. [13] have independently developed a Cubical Agda formalization of the HoTT book reals, pursued as an in-progress pull request to the Cubical Agda standard library. They additionally contribute algebraic and premetric infrastructure, including pseudolattices, ordered commutative rings, and Archimedean rings adapted from the HoTT book's ordered Heyting fields. In ongoing work, they generalize Gilbert's premetric spaces from positive rationals to the positive cone of an arbitrary ordered commutative ring. Their goals extend into analysis beyond the scope of this thesis, including the Riemann integral, the fundamental theorem of calculus, the mean value theorem, and trigonometric functions.

The HoTT book construction can be read as an adaptation of earlier work by Richman [14], who developed a notion of completion for premetric spaces that does not require countable choice. The HoTT book's closeness relation and limit constructor apply Richman's completion strategy to the rationals within univalent foundations.

Booij [8] pursues constructive analysis in univalent type theory. Booij surveys several constructions of the real numbers, including an alternative characterization of the HoTT book reals as a homotopy-initial Cauchy structure, and proves equivalences among them. Booij's central contribution is the notion of a *locator*, extra structure attached to reals that enables discrete observations such as signed-digit expansions while remaining compatible with the extensionality principles of univalent type theory. Using locators, Booij develops intermediate value theorems, Riemann integration, and the Picard-Lindelöf theorem. The work is pencil-and-paper mathematics accompanied by a Haskell prototype of locators. A chapter addresses strategies for proof-assistant implementation but stops short of formalization. We return to locators in Chapter 5, where we discuss generalizing them beyond the real numbers to a broader class of structures.

Murray [15] formalized Bishop-style constructive real numbers in non-cubical Agda, including their arithmetic, ordering, Cauchy completeness, and uncountability. In his future work, Murray suggested a Cubical Agda port as a possible next step, while noting that HoTT-based definitions of the real numbers might be preferable to Bishop-style ones, rendering such a port redundant. Our work here takes up exactly that suggestion, replacing the Bishop-style setoid construction with the HoTT book reals as a higher inductive-inductive type in Cubical Agda.

5 - FUTURE WORK

We identify three directions for future work arising from this formalization, concerning the algebraic packaging of the HoTT book reals, the extension principles used to lift operations from the rationals, and the generalization of locators to a broader class of structures.

Our formalization stops just short of characterizing the HoTT book reals under their universal property, namely that they are the initial Cauchy complete Archimedean ordered field. We cover up to the end of Section 11.3.3 of the HoTT book, concluding with Theorem 11.3.48 which states that the HoTT book reals form an Archimedean ordered field [1]. Section 11.3.4 comprises only two further theorems: Theorem 11.3.49 shows Cauchy completeness and Theorem 11.3.50 shows initiality. We stopped short not because these theorems are difficult to prove, but because the Cubical Agda standard library lacked support for a constructive formulation of fields at the time of writing. As a consequence, our formalization does not contain a unified statement corresponding to Theorem 2.4.26. Instead, our results are scattered across each of the properties which make up an Archimedean ordered field definition, although we have verified each constituent property individually, so that the proof will follow immediately once a suitable definition is in place. A natural next step would be to add these definitions to the Cubical Agda standard library, allowing the universal property to be stated formally.

We mentioned in Section 2.3 that the Lipschitz and non-expanding extension principles of Lemmas 2.3.1 and 2.3.2 were not to be read as the final word on the conditions under which maps on the rationals extend to the reals. The constructions of multiplication and reciprocal in Section 2.4 required iterated Lipschitz extensions followed by elimination of the chosen bound via Kraus's theorem, adding considerable complexity compared to the constructions of negation, addition, and the lattice operations. A broader extension principle for uniformly continuous maps would simplify at least the local extension step, since bounded multiplication is uniformly continuous as a map of two rational variables. Booij suggests that such an extension ought to be possible [8, Chapter 10].

The path constructor for the HoTT book reals ensures that two reals which are arbitrarily close are identified, so that the natural notion of equality among reals coincides with the identity type. This avoids the drawbacks of a Bishop-style setoid construction, where the equivalence relation and the identity type diverge. However, this identification is also exactly what prevents us from extracting rational approximations. Booij addresses this by introducing the concept of a locator, extra structure on a real number that restores the ability to compute with approximations [8]. A locator is obtained by strengthening the locatedness property of Dedekind cuts from a property to structure. At the end of his chapter on metric spaces, Booij writes: "We have not found a definition of locators for general metric spaces, of which the locators introduced in Chapter 6 would be a special case. This would be an important future direction of research" [8, p. 183]. We have developed a construction of the HoTT book reals in Cubical Agda, where our proofs are type-checked for correctness. Generalizing locators to a larger class of structures would allow the same framework that verifies the correctness of the mathematical theory to also execute computations on analytic objects via term normalization.

6 - CONCLUSION

This thesis presented a formalization of the HoTT book reals in Cubical Agda. We covered the higher inductive-inductive definition, the closeness relation and its alternative characterization, extension principles for Lipschitz and non-expanding maps, and the algebraic and order-theoretic structure culminating in the proof that the HoTT book reals form an Archimedean ordered field. The development is open source at `https://github.com/utahplt/hott-reals` and typechecks without postulates or holes.

The process of formalization required us to make explicit mathematical structure that the informal presentation of the HoTT book left implicit, including the precise form of the enhanced recursion principle and the hypotheses needed for multi-variable continuous extensions. In several cases, the precision demanded by the type checker surfaced insights that were not apparent from reading the informal account alone.

This development clarifies how the HoTT book reals behave in a proof assistant with native support for higher inductive types. It provides a foundation for further machine-assisted work in constructive analysis.

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