A note on the Lovász theta number of random graphs

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Abstract

We show a strong concentration bound for the Lovász ϑ function on $G(n,p)$ random graphs. For $p = 1/2$, for instance, our result implies that the ϑ function is concentrated in an interval of length polylog(n) w.h.p. The best known bound previously was roughly $n^{1/4}$. The general idea is to prove that all the vectors in an optimal solution have "roughly equal lengths" w.h.p.

1 Introduction

The Lovász ϑ function of a graph is a quantity introduced by Lovász to study the Shannon capacity of a graph [4]. It is a semidefinite programming relaxation for the independent set of a graph. For a graph $G = (V, E)$, it is formally defined as follows (see [4] for other equivalent formulations)

$$\begin{aligned}\vartheta(G) := \max \sum_i v_i \cdot v_0 \quad &\text{s.t.}\\ v_i^2 = v_i \cdot v_0 \quad &\forall i\\ v_0^2 = 1&\\ \langle v_i, v_j \rangle = 0 \quad &\forall \{i,j\} \in E(G)\end{aligned}$$

The expected value of the Lovász ϑ function for $G(n,p)$ random graphs was first studied by Juhász [3], who showed that for $G \sim G(n,p)$ and $p \geq \log^2 n/n$, we have

$$\sqrt{\frac{n}{p}} \leq \vartheta(G) \leq 2\sqrt{\frac{n}{p}} \quad \text{w.p. at least } 1 - \frac{1}{n}.$$

More recently, Coja-Oghlan studied the concentration properties of the ϑ function for $G(n,p)$ random graphs [2]. He proved that the ϑ function is concentrated in intervals of length $O(1)$ w.h.p. when $p < n^{-1/2}$. More precisely, he proves the following large deviation bound for $\vartheta(G)$: suppose $G \sim G(n,p)$ and let μ be the median value of $\vartheta(G)$. Then

$$\Pr[|\vartheta(G) - \mu| > t] \leq e^{-t^2/(\mu+t)}.$$

Note that for say $p = 1/2$, this only says that $\vartheta(G)$ is concentrated in an interval of length roughly $n^{1/4}$ w.h.p.[1] In this note, we will show a better tail bound. More precisely,

Theorem 1. *Let $G = (V, E)$ be a graph drawn from $G(n, 1/2)$. Let μ denote the median of $\vartheta(G)$ for this distribution. Then for some absolute constant C, we have*

$$\Pr[|\vartheta(G) - \mu| > t] \leq e^{-t^{4/3}/(C \log^3 n)}, \tag{1}$$

[1] Throughout, when we say "w.h.p.", we mean w.p. at least $1 - \frac{1}{n^c}$ for any constant c (there will be certain parameters which naturally depend on c).

Our techniques are not specific to $p = 1/2$, but for ease of exposition, we will only work with this case. This implies, for instance, that for $G(n, 1/2)$ random graphs, $\vartheta(G)$ is concentrated in intervals of size only polylog(n).

Comment. The exponent 4/3 is unnatural, and we believe it is an artefact of our proof – we conjecture that the "true" tail bound is in fact (1) with $e^{-t^2/C\log n}$ on the RHS.

2 Proof

In what follows, let μ denote the median of $\vartheta(G)$ for $G \sim G(n, 1/2)$, and let t be a given parameter. Let s be a parameter (we will set it to be $\max\{t^{2/3}, \log n\}$). A graph G is said to be s-bad if for all vector solutions v_i which "realize" the optimum value for the relaxation $\vartheta(G)$, we have

$$\sum_{i\in V} \|v_i\|^4 > (1+s)\log^2 n.$$

Lemma 2. *Suppose G is s-bad for some $s \geq \log n$. Then there exists an $S \subseteq V$ of size $k \geq s$ such that the induced subgraph H on S has $\vartheta(H) > \sqrt{k(1+s)\log n}$.*

Proof. Let $\{v_i\}_{i=1}^n$ denote an optimum vector solution for the ϑ relaxation on G. It is easy to see that there exists a solution with value at least $\vartheta(G)/2$ and the additional property $\|v_i\|^2 \geq \frac{1}{2n}$ (we can simply set vectors which are smaller than this length to zero). Now divide the v_i into $\log n$ levels based on $\|v_i\|^2$, such that the value of $\|v_i\|^2$ varies by a factor at most 2 in each level.

Since G is s-bad, we have that $\sum_i \|v_i\|^4 \geq (1+s)\log^2 n$. There exists a level which contributes at least a $1/\log n$ fraction to the sum: let S be the set of indices in this level, and let $k = |S|$. Thus for each $i \in S$, we have $\|v_i\|^2 \approx \left(\frac{(1+s)\log n}{k}\right)^{1/2}$, implying that $\sum_{i\in S} v_i^2 \geq \frac{1}{2}\sqrt{k(1+s)\log n}$. Since v_i is a feasible solution to the relaxation $\vartheta(G)$, it is clear that the restriction to S gives a feasible solution to $\vartheta(H)$. Thus $\vartheta(H) \geq \sqrt{k(1+s)\log n}$.

Finally, since $\|v_i\|^2 \leq 1$, we must have $k \geq s$, thus proving the lemma. □

We can now bound the probability that $G \sim G(n, 1/2)$ is s-bad for some $s \geq \log n$. Fix some set $S \subseteq V$ of size k and let H be the induced subgraph on S in G. We now use a bound of [4] relating $\vartheta(H)$ to the eigenvalues of its adjacency matrix.

Lemma 3. *[4] Let G be a graph with adjacency matrix $A(G)$, J denote the $n \times n$ matrix of ones, and I the identity matrix. Then*

$$\vartheta(G) \leq \lambda_{\max}(J - 2A(G) - I).$$

We refer to the paper of Lovász for the proof [4]. It follows from one of the equivalent definitions of the ϑ function. The second ingredient is a concentration bound for the top eigenvalues of a random matrix due to Alon, Krivelevich and Vu [1]. They prove the following.

Lemma 4. *Let A be a symmetric $n \times n$ matrix with the upper diagonal entries drawn i.i.d. from a distribution with mean zero and variance 1. Then for all $t > 0$, and integer $r \geq 1$, we have*

$$\Pr[|\lambda_r(A) - \mu(\lambda_r(A))| \geq t] \leq e^{-t^2/2r^2}. \quad (2)$$

(As usual λ_r denotes the rth largest eigenvalue, and $\mu(\lambda_r)$ denotes the median of this value over the distribution)

Now we note that for any fixed $S \subseteq V$ of size k, the matrix $J - 2A(H) - I$ is a $k \times k$ symmetric matrix with entries being i.i.d. ± 1 (and zero on the diagonal). Thus the median of $\lambda_{\max}(J - 2A(G) - I)$ is at most $(2 + o(1))\sqrt{k}$, and by Lemma 4, we have

$$\Pr\left[\lambda_{\max}(A(H)) > \sqrt{k(1+s)\log n}\right] < e^{-k(1+s)\log n}.$$

Now by Lemma 3, the probability that $\vartheta(H) > \sqrt{k(1+s)\log n}$ is also bounded by the same quantity. Thus we can take a union bound over all subsets of size $k \geq s$, and by Lemma 2, we have

$$\Pr[G \text{ is } s\text{-bad}] \leq \sum_{k \geq s} \binom{n}{k} \cdot e^{-(1+s)k \log n} < \sum_{k \geq s} e^{-sk \log n} \leq e^{-s^2 \log n}.$$

(In the above we used $k \geq s$, and a simple bound on $\binom{n}{k}$). We have thus proved that

Lemma 5. *Let $G \sim G(n, 1/2)$, and $s \geq \log n$. The probability that G is s-bad is at most $e^{-s^2 \log n}$.*

We can now follow the proof of Coja-Oghlan [2] (and [1]) and use Talagrand's inequality. Let us first recall it.

Theorem 6. *(Talagrand)[5] Let Ω be a set with a measure μ defined on it, and let $A, B \subseteq \Omega^n$. Let μ_n denote the* product measure *obtained from μ. Suppose A and B are "t-separated" in the following way: for every $b \in B$, there exist weights $\{\alpha_i\}_{i=1}^n$ with $\sum_i \alpha_i^2 \leq 1$ such that*

$$\forall\ a \in A, \quad \sum_{i: a_i \neq b_i} \alpha_i \geq t.$$

Then we have $\mu_n(A)\mu_n(B) \leq e^{-t^2}$.

The theorem is very powerful, and we typically use it with finite sets Ω. Let us now define two sets of graphs as follows

$\mathcal{A} := \{G : \ \vartheta(G) \leq \mu\}$, and

$\mathcal{B} := \{G : \ \vartheta(G) \geq \mu + t$, and G is not s-bad for $s = \max\{t^{2/3}, \log n\}\}$.

Let $m(\mathcal{A})$ (similarly $\mathcal{B}$) denote the measure of $\mathcal{A}$ in the set of graphs $G(n, 1/2)$. Since μ was defined to be the median, $m(\mathcal{A}) = 1/2$.

Lemma 7.

$$m(\mathcal{A}) \cdot m(\mathcal{B}) \leq e^{-t^2/(1+s)\log n}\ .$$

Proof. Consider a graph $B \in \mathcal{B}$. Let $\{v_i\}_{i=1}^n$ be the set of vectors in an optimal solution to the ϑ-relaxation on B. Now consider any $A \in \mathcal{A}$.

Let α_i be 1 if vertex i has precisely the same set of neighbors in both A and B, and 0 otherwise. Now observe that $\{\alpha_i v_i\}$ is a feasible vector solution to the ϑ relaxation for B (because $\alpha_i \alpha_j \neq 0$ implies $\{i, j\}$ is an edge in B iff it is an edge in A). Thus $\sum_i (\alpha_i v_i)^2 \leq \vartheta(B)$, hence $\sum_{i:\Gamma_A(i) \neq \Gamma_B(i)} v_i^2 \geq t$ (since $\vartheta(B) < \mu$).

Now by the definition of $\mathcal{B}$, B is not s-bad, hence we have $\sum_i (v_i^2)^2 \leq (1+s)\log^2 n$. By Talagrand's inequality,[2] we have

$$m(\mathcal{A}) \cdot m(\mathcal{B}) \leq e^{-t^2/(1+s)\log^2 n}\ .$$

□

[2]Formally, the product space here is Ω^n, where Ω consists of vectors in $\{0, 1\}^n$ representing the adjacency vectors of a vertex in the graph. In these terms, α_i is an indicator for the ith vectors corresponding to A, B being equal.

Corollary 8. *Let $G \sim G(n, 1/2)$. Then*

$$\Pr[\vartheta(G) > \mu + t] \leq e^{-t^{4/3}/\log^2 n}.$$

Proof. From the above lemmas, we can bound the desired probability by

$$\begin{aligned}&\Pr[G \text{ is } s\text{-bad}] + \Pr[G \in \mathcal{B}] \\ &\quad = e^{-s^2} + e^{-t^2/(1+s)\log^2 n} \leq e^{-t^{4/3}/\log^3 n}.\end{aligned}$$

The last inequality is due to our choice of s. □

Lower tail. A bound for the lower tail is actually easier to prove: as before, define two sets

$$\begin{aligned}\mathcal{A} &:= \{G : \vartheta(G) \leq \mu - t\}, \text{ and} \\ \mathcal{B} &:= \{G : \vartheta(G) \geq \mu, \text{ and } G \text{ is not } \log n\text{-bad}\}.\end{aligned}$$

The key is to note that the probability that G is $\log n$-bad is only $e^{-\log^2 n} \ll 1/10$, and thus $m(\mathcal{B}) \geq 1/3$ (because without this restriction, the measure is $1/2$, since μ is the median). Now using precisely the same argument as above, we obtain

$$m(\mathcal{A}) \leq e^{-t^2/\log^3 n}.$$

This completes the proof of Theorem 1.

References

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