Homework 2 (Due Thursday, Feb 4 (23:59 MT))

Policy: You are allowed to reference any material, but please cite the source and write the solutions in your own words. Discussing solutions is OK, as long as you mention the names of students you discussed with.

1. (10 points) A 3-SAT formula on n variables with m clauses is the following: we have n boolean variables $x_1, x_2, \ldots, x_n$, and m *clauses* $C_1, C_2, \ldots, C_m$. Each C_i is the boolean *OR* of three distinct literals (a literal is just a variable x_i or its negation $\neg x_i$). An example of a clause is $x_2 \vee x_4 \vee \neg x_5$. An assignment $\sigma \in \{T, F\}^n$ to the x_j's is said to *satisfy* clause C_i if one of the literals in the clause is set to TRUE. (E.g., to satisfy the clause above, we must have either x_2 or x_4 to be TRUE, or x_5 to be FALSE.)

   Show that for any 3-SAT formula with m clauses, there exists an assignment that satisfies at least $7m/8$ clauses.

2. (10 points) A *set system* $\mathcal{F}$ over the set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets $S_1, S_2, \ldots, S_m$ of $[n]$. Our goal is to *color* the integers $\{1, 2, \ldots, n\}$ with two colors (say $-1, 1$), such that each of the sets S_i is as *balanced* as possible. Formally, if we have a coloring $\chi : [n] \mapsto \{-1, 1\}$, the *imbalance* (or discrepancy) is defined as

$$\max_{1 \le i \le m} \left| \sum_{j \in S_i} \chi(j) \right| .$$

   Prove that for any set system $\mathcal{F}$ with m sets, there exists a coloring with imbalance at most $O(\sqrt{n \log m})$.

3. (10 points) Let G be a connected d-regular graph, and consider its adjacency matrix A_G. Prove that $\lambda_{\min}(A_G) = -d$ if and only if G is bipartite.

4. (10 points) Let G be a connected d-regular graph, and suppose the Laplacian L_G has eigenvalues $0 \le \lambda_2 \le \cdots \le \lambda_n$. Then, without using Cheeger's inequality, prove that $\lambda_2 \ge \frac{1}{4n^2}$.

   HINT: recall that $\lambda_2 = \min_{\|x\|=1, \sum_i x_i = 0} \sum_{\{i,j\} \in E} (x_i - x_j)^2$; prove that there exists an x_u, x_v such that $|x_u - x_v| \ge 1/\sqrt{n}$, and consider the path from u to v.

5. (10 points) Let G be a d-regular graph with the property that *every subset* S of size at most $n/2$ has expansion at least $1/4$, i.e.,

$$\frac{E(S, V \setminus S)}{d|S|} \ge \frac{1}{4}.$$

   (a) (3 points) Define the *neighborhood* $N(S)$ of a set S to be the set of all vertices $u \in V \setminus S$ that have at least one edge to a vertex in S. Prove that for any V of size at most $n/2$ in our graph G,

$$|N(S)| \ge |S|/4.$$

(b) (7 points) Prove that the *diameter* of the graph is $O(\log n)$. I.e., for any $u, v \in V$, prove that there exists a path of length at most $O(\log n)$ between u and v.

HINT: start with u, and consider the number of vertices at a distance ℓ from u; inductively do this for $\ell = 1, 2, \ldots$; for $\ell = O(\log n)$, argue that the number is $\geq n/2$; do the same with v.