Algorithms, Geometry, and Optimization

(Lecture 21: Mon, Apr 1 2024)

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Volume estimation

**Problem.** Given an $n$-dimensional convex body $K$ (in the form of *membership* oracle), estimate $\text{vol}(K)$.

**Surprising:** deterministic algorithms cannot do this!

*Theorem* (last lecture): For any deterministic algo that makes $< 2^{n/2}$ queries, there is an instance where the estimation is off by $2^{n/2}$. 
Randomized algorithms?

Can obtain $(1 + \epsilon)$ approximation with only polynomial \((n, \frac{1}{\epsilon})\) many queries to the oracle.

**Assumption.** (technical) for some known \(R\),

\[ \mathcal{B} \subseteq K \subseteq R\mathcal{B}. \]

(Runtime includes a \(\log R\) factor.)
Outline

Claim 1. Membership oracle \(\implies\) "sampling" oracle (poly time, obtain a random sample from \(K\))

\textit{Spoiler:} done via random walks!

Claim 2. This implies efficient volume estimation

One weird trick...
Sampling & volume finding

Example: blindfold dartboards

What is the probability that dart lands on a black colored region?
Curse of dimensionality

Volume of a cube / volume of "minimum enclosing" sphere is \( \exp(-n) \)

\[ \rightarrow \text{need to throw } \exp(n) \text{ darts!} \]

[Dyer, Frieze, Kannan] trick: define

\[ K_j := \left( 1 + \frac{1}{n} \right)^j \mathcal{B} \cap K \]

By definition, \( K_0 = \mathcal{B}, \ K_m = K \), for \( m \approx n \log R \)
Can write:

\[
\frac{\text{vol}(K)}{\text{vol}(K_0)} = \frac{\text{vol}(K_m)}{\text{vol}(K_{m-1})} \cdot \frac{\text{vol}(K_{m-1})}{\text{vol}(K_{m-2})} \cdots \frac{\text{vol}(K_1)}{\text{vol}(K_0)}.
\]

**Key idea.** Each of these terms is between \([1, 3]\).

Thus, to estimate to error \((1 + \gamma)\), need \(\approx \frac{1}{\gamma^2}\) samples.

Choose \(\gamma = \frac{\epsilon}{2m}\) (\(\epsilon\) is desired overall accuracy)
Recall outline

Claim 1. Membership oracle $\implies$ "sampling" oracle (poly time, obtain a random sample from $K$)

[Spoiler: done via random walks!]

Claim 2. This implies efficient volume estimation

One weird trick...
Sampling from membership

How to create random sample from $K$?

- Start with any point $x_0$
- Pick random point $z$ in $\text{Ball}(x_0, \delta)$ for some $\delta > 0$
- If $z$ is in $K$ (membership), set $x_1 = z$, else $x_1 = x_0$
- Repeat, $x_2, x_3, \ldots$ for $N = \text{poly}(n)$ steps
Problems

1. "Needles" -- can be inefficient -- but convex objects don't have too many!

2. What is the stationary distribution of this random walk?

3. How many iterations? main challenge