Algorithms, Geometry, and Optimization

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Volume estimation

Problem. Givien an n-dimensional convex body K (in the form of *membership* oracle), estimate $\text{vol}(K)$.

Surprising: deterministic algorithms cannot do this!

Theorem (last lecture): For any deterministic algo that makes $< 2^{n/2}$ queries, there is an instance where the estimation is off by $2^{n/2}$.

Randomized algorithms?

Can obtain $(1+\epsilon)$ approximation with only polynomial $(n, \frac{1}{\epsilon})$ many queries to the oracle.

Assumption. (technical) for some known R,

$$\mathcal{B} \subseteq K \subseteq R\mathcal{B}.$$

(Runtime includes a $\log R$ factor.)

Outline

Claim 1. Membership oracle $\implies$ "sampling" oracle
(poly time, obtain a random sample from K)

[*Spoiler:* done via random walks!]

Claim 2. This implies efficient volume estimation

One weird trick...

Sampling & volume finding

Example: blindfold dartboards

What is the probability that dart lands on a black colored region?

Curse of dimensionality

Volume of a cube / volume of "minimum enclosing" sphere is $\exp(-n)$

$\implies$ need to throw $\exp(n)$ darts!

[Dyer, Frieze, Kannan] trick: define

$$K_j := \left(1 + \frac{1}{n}\right)^j \mathcal{B} \cap K$$

By definition, $K_0 = \mathcal{B}$, $K_m = K$, for $m \approx n \log R$

DFK trick (contd.)

Can write:

$$\frac{\mathrm{vol}(K)}{\mathrm{vol}(K_0)} = \frac{\mathrm{vol}(K_m)}{\mathrm{vol}(K_{m-1})} \cdot \frac{\mathrm{vol}(K_{m-1})}{\mathrm{vol}(K_{m-2})} \cdots \frac{\mathrm{vol}(K_1)}{\mathrm{vol}(K_0)}.$$

Key idea. Each of these terms is between $[1, 3]$.

Thus, to estimate to error $(1+\gamma)$, need $\approx \frac{1}{\gamma^2}$ samples.

Choose $\gamma = \frac{\epsilon}{2m}$ (ϵ is desired overall accuracy)

Recall outline

Claim 1. Membership oracle $\implies$ "sampling" oracle (poly time, obtain a random sample from K)

[*Spoiler:* done via random walks!]

Claim 2. This implies efficient volume estimation

One weird trick...

Sampling from membership

How to create random sample from K?

- Start with any point x_0
- Pick random point z in $\text{Ball}(x_0, \delta)$ for some $\delta > 0$
- If z is in K (membership), set $x_1 = z$, else $x_1 = x_0$
- Repeat, $x_2, x_3, \ldots$ for $N = \text{poly}(n)$ steps

Problems

1. "Needles" -- can be inefficient -- but convex objects don't have too many!

2. What is the stationary distribution of this random walk?
3. How many iterations? **main challenge**