Detail-preserving Mesh Unfolding for Non-rigid Shape Retrieval - Supplementary material

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1 Closed forms

The relatively simple closed forms of the Jacobian and Hessian of our constraints lead to efficient solutions with any solver of choice. All the derivations are based on the following volume calculation of tetrahedron t consisting of vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbf{v}$.

$$\begin{aligned}\mathrm{vol}(t, \mathbf{v}) = [&\mathbf{v}_2^x\mathbf{v}_3^y\mathbf{v}_4^z - \mathbf{v}_2^x\mathbf{v}_4^y\mathbf{v}_3^z - \mathbf{v}_1^x\mathbf{v}_2^y\mathbf{v}_3^z + \mathbf{v}_1^x\mathbf{v}_2^y\mathbf{v}_4^z - \mathbf{v}_1^x\mathbf{v}_4^y\mathbf{v}_2^z - \mathbf{v}_1^x\mathbf{v}_3^y\mathbf{v}_4^z +\\ &\mathbf{v}_1^x\mathbf{v}_3^y\mathbf{v}_2^z + \mathbf{v}_1^x\mathbf{v}_4^y\mathbf{v}_3^z - \mathbf{v}_3^x\mathbf{v}_4^y\mathbf{v}_1^z + \mathbf{v}_3^x\mathbf{v}_4^y\mathbf{v}_2^z + \mathbf{v}_3^x\mathbf{v}_1^y\mathbf{v}_4^z - \mathbf{v}_3^x\mathbf{v}_2^y\mathbf{v}_4^z -\\ &\mathbf{v}_2^x\mathbf{v}_3^y\mathbf{v}_1^z + \mathbf{v}_2^x\mathbf{v}_4^y\mathbf{v}_1^z + \mathbf{v}_2^x\mathbf{v}_1^y\mathbf{v}_3^z - \mathbf{v}_2^x\mathbf{v}_1^y\mathbf{v}_4^z + \mathbf{v}_4^x\mathbf{v}_1^y\mathbf{v}_2^z - \mathbf{v}_4^x\mathbf{v}_1^y\mathbf{v}_3^z +\\ &\mathbf{v}_4^x\mathbf{v}_2^y\mathbf{v}_3^z - \mathbf{v}_4^x\mathbf{v}_2^y\mathbf{v}_1^z + \mathbf{v}_4^x\mathbf{v}_3^y\mathbf{v}_1^z - \mathbf{v}_4^x\mathbf{v}_3^y\mathbf{v}_2^z - \mathbf{v}_3^x\mathbf{v}_1^y\mathbf{v}_2^z + \mathbf{v}_3^x\mathbf{v}_2^y\mathbf{v}_2^z -\\ &\mathbf{v}_3^x\mathbf{v}_2^y\mathbf{v}_3^z + \mathbf{v}_3^x\mathbf{v}_2^y\mathbf{v}_1^z]/6,\end{aligned}$$

where $\mathbf{v}_i^x, \mathbf{v}_i^y$, and $\mathbf{v}_i^z$ are the x, y, and z coordinates of the vertex $\mathbf{v}_i$, respectively.

1.1 First derivatives for Jacobian

1.1.1 Constraint: $c_t(\mathbf{v}) : \mathrm{vol}(t) > \epsilon \quad \forall t \in T$

Constraint on the tetrahedron t consisting of vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbf{v}$:

$$\nabla c_t(\mathbf{v}) = \begin{bmatrix} \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_1^x \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_1^y \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_1^z \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_2^x \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_2^y \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_2^z \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_3^x \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_3^y \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_3^z \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_4^x \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_4^y \\ \partial\mathrm{vol}(t,\mathbf{v})/\partial\mathbf{v}_4^z \end{bmatrix} = \begin{bmatrix} ((-\mathbf{v}_2^y + \mathbf{v}_3^y)(\mathbf{v}_3^z - \mathbf{v}_4^z) + (\mathbf{v}_3^y - \mathbf{v}_4^y)(\mathbf{v}_2^z - \mathbf{v}_3^z))/6 \\ ((\mathbf{v}_3^x - \mathbf{v}_2^x)(-\mathbf{v}_3^z + \mathbf{v}_4^z) + (\mathbf{v}_4^x - \mathbf{v}_3^x)(\mathbf{v}_2^z - \mathbf{v}_3^z))/6 \\ ((\mathbf{v}_3^x - \mathbf{v}_2^x)(\mathbf{v}_3^y - \mathbf{v}_4^y) + (\mathbf{v}_4^x - \mathbf{v}_3^x)(-\mathbf{v}_2^y + \mathbf{v}_3^y))/6 \\ A \\ B \\ C \\ D \\ E \\ F \\ ((\mathbf{v}_1^y - \mathbf{v}_2^y)(\mathbf{v}_2^z - \mathbf{v}_3^z) - (\mathbf{v}_2^y - \mathbf{v}_3^y)(\mathbf{v}_1^z - \mathbf{v}_2^z))/6 \\ ((\mathbf{v}_2^x - \mathbf{v}_1^x)(\mathbf{v}_2^z - \mathbf{v}_3^z) + (\mathbf{v}_3^x - \mathbf{v}_2^x)(-\mathbf{v}_1^z + \mathbf{v}_2^z))/6 \\ ((\mathbf{v}_2^x - \mathbf{v}_1^x)(-\mathbf{v}_2^y + \mathbf{v}_3^y) + (\mathbf{v}_3^x - \mathbf{v}_2^x)(\mathbf{v}_1^y + \mathbf{v}_2^y))/6 \end{bmatrix},$$

where the rows that do not fit inside the matrix layout are as follows:
$A = ((\mathbf{v}_2^y - \mathbf{v}_3^y)(\mathbf{v}_3^z - \mathbf{v}_4^z) - (\mathbf{v}_3^y - \mathbf{v}_4^y)(\mathbf{v}_2^z - \mathbf{v}_3^z) - (\mathbf{v}_3^y - \mathbf{v}_4^y)(\mathbf{v}_1^z - \mathbf{v}_2^z) + (\mathbf{v}_1^y - \mathbf{v}_2^y)(\mathbf{v}_3^z - \mathbf{v}_4^z))/6$,
$B = ((\mathbf{v}_2^x - \mathbf{v}_1^x)(\mathbf{v}_3^z - \mathbf{v}_4^z) + (\mathbf{v}_3^x - \mathbf{v}_2^x)(\mathbf{v}_3^z - \mathbf{v}_4^z) + (\mathbf{v}_4^x - \mathbf{v}_3^x)(\mathbf{v}_3^z - \mathbf{v}_1^z))/6$,
$C = ((\mathbf{v}_2^x - \mathbf{v}_1^x)(-\mathbf{v}_3^y + \mathbf{v}_4^y) + (\mathbf{v}_3^x - \mathbf{v}_2^x)(-\mathbf{v}_3^y + \mathbf{v}_4^y) + (\mathbf{v}_4^x - \mathbf{v}_3^x)(\mathbf{v}_1^y - \mathbf{v}_3^y))/6$,
$D = ((\mathbf{v}_3^y - \mathbf{v}_4^y)(\mathbf{v}_1^z - \mathbf{v}_2^z) - (\mathbf{v}_1^y - \mathbf{v}_2^y)(\mathbf{v}_3^z - \mathbf{v}_4^z) - (\mathbf{v}_1^y - \mathbf{v}_2^y)(\mathbf{v}_2^z - \mathbf{v}_3^z) + (\mathbf{v}_2^y - \mathbf{v}_3^y)(\mathbf{v}_1^z - \mathbf{v}_2^z))/6$,
$E = ((\mathbf{v}_2^x - \mathbf{v}_1^x)(\mathbf{v}_4^z - \mathbf{v}_2^z) + (\mathbf{v}_3^x - \mathbf{v}_2^x)(\mathbf{v}_1^z - \mathbf{v}_2^z) + (\mathbf{v}_4^x - \mathbf{v}_3^x)(\mathbf{v}_1^z - \mathbf{v}_2^y))/6$,
$F = ((\mathbf{v}_2^x - \mathbf{v}_1^x)(\mathbf{v}_2^y - \mathbf{v}_4^y) + (\mathbf{v}_3^x - \mathbf{v}_2^x)(-\mathbf{v}_1^y + \mathbf{v}_2^y) + (\mathbf{v}_4^x - \mathbf{v}_3^x)(-\mathbf{v}_1^y + \mathbf{v}_2^y))/6$.

1.1.2 Constraint: $c_i(\mathbf{v}) : \sum_{t \in \eta(i)} \mathrm{vol}(t) = l_i \quad \forall i \in V$

Constraint on the vertex $\mathbf{v}_i$ that is adjacent to tetrahedra in the set $\eta(i)$:

$$\nabla c_i(\mathbf{v}) = \sum_{t \in \eta(i)} \nabla c_t(\mathbf{v}) = \sum_{t \in \eta(i)} \begin{bmatrix} \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^x \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^y \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^z \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^x \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^y \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^z \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^x \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^y \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^z \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^x \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^y \\ \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^z \end{bmatrix},$$

where the rows are simply the copies of the corresponding 12 rows in Sec. 1.1.1.

1.2 Second derivatives for Hessian

1.2.1 Constraint: $c_t(\mathbf{v}) : \mathrm{vol}(t) > \epsilon \quad \forall t \in T$

Constraint on the tetrahedron t consisting of vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbf{v}$:

$$\nabla^2 c_t(\mathbf{v}) = \begin{bmatrix}
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^x \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^x \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^x \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^y \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^y \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^y \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^z \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^z \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_1^z \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^x \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^x \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^x \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^y \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^y \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^y \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^z \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^z \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_2^z \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^x \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^x \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^x \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^y \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^y \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^y \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^z \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^z \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_3^z \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^x \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^x \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^x \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^y \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^y \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^y \partial \mathbf{v}_4^z \\
\partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^z \partial \mathbf{v}_1^x & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^z \partial \mathbf{v}_1^y & \dots & \partial \mathrm{vol}(t,\mathbf{v})/\partial \mathbf{v}_4^z \partial \mathbf{v}_4^z
\end{bmatrix},$$

where we provide the expansion of one entry only; all others can be expanded similarly. Note that some of the 12 entries in each row may be equal to 0; e.g., $\partial \mathrm{vol}(t, \mathbf{v})/\partial \mathbf{v}_1^x \partial \mathbf{v}_1^y = 0$ as $\partial \mathrm{vol}(t, \mathbf{v})/\partial \mathbf{v}_1^x = ((-\mathbf{v}_2^y + \mathbf{v}_3^y)(\mathbf{v}_3^z - \mathbf{v}_4^z) + (\mathbf{v}_3^y - \mathbf{v}_4^y)(\mathbf{v}_2^z - \mathbf{v}_3^z))/6$ (from Sec. 1.1.1) does not have any term with $\mathbf{v}_1^y$. The entry $\partial \mathrm{vol}(t, \mathbf{v})/\partial \mathbf{v}_1^x \partial \mathbf{v}_4^z$, on the other hand, is equal to $(\mathbf{v}_2^y - \mathbf{v}_3^y)/6$.

1.2.2 Constraint: $c_i(\mathbf{v}) : \sum_{t \in \eta(i)} \mathrm{vol}(t) = l_i \quad \forall i \in V$

Constraint on the vertex $\mathbf{v}_i$ that is adjacent to tetrahedra in the set $\eta(i)$:

$$\nabla^2 c_i(\mathbf{v}) = \sum_{t \in \eta(i)} \nabla^2 c_t(\mathbf{v}),$$

where $\nabla^2 c_t(\mathbf{v})$ is given in Sec. 1.2.1.

As far as the objective function $E(\mathbf{v})$ is concerned, the standard first and second derivatives of the mass-spring system are used [Nealen et al. 2006].