Supplemental Material for "Fast and Robust Inversion-Free Shape Manipulation"

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1 Relation between semidefinite constraint and noninversion

In our paper we replace the noninversion constraint $\det(\mathbf{F}_i) > 0$ with the semidefiniteness constraint $\mathbf{S}_i \succ 0$, where $\mathbf{S}_i = \text{sym}\{\hat{\mathbf{R}}_i\mathbf{F}_i\}$. (Note that $\text{sym}\{\mathbf{M}\} = \frac{1}{2}(\mathbf{M} + \mathbf{M}^T)$ denotes the symmetric part of matrix $\mathbf{M}$.)

In this section we show that the semidefinite constraint subsumes the positivity of the determinant, i.e. $\mathbf{S}_i \succ 0 \Rightarrow \det(\mathbf{F}_i) > 0$, and show bounds for the determinant that can be expressed using $\mathbf{S}_i$.

Lemma 1. *Let $\mathbf{\Omega} \in \mathbb{C}^{n\times n}$ be a skew-hermitian matrix, i.e. $\mathbf{\Omega}^* = -\mathbf{\Omega}$ where $\mathbf{\Omega}^*$ denotes the conjugate transpose of $\mathbf{\Omega}$. Then all eigenvalues of $\mathbf{\Omega}$ are imaginary (or zero).*

Proof. Let $(\mathbf{q}, \lambda)$ be an eigenvector-eigenvalue pair. Then

$$\mathbf{\Omega q} = \lambda q \Rightarrow \mathbf{q}^*\mathbf{\Omega q} = \lambda \mathbf{q}^*\mathbf{q} = \lambda||\mathbf{q}||^2$$

Taking the conjugate transpose of the equation above, we have

$$(\mathbf{q}^*\mathbf{\Omega q})^* = (\lambda||\mathbf{q}||^2)^* \Rightarrow \mathbf{q}^*\mathbf{\Omega}^*\mathbf{q} = \lambda^*||\mathbf{q}||^2 \Rightarrow -\mathbf{q}^*\mathbf{\Omega q} = \lambda^*||\mathbf{q}||^2$$

Adding the two equations $\lambda||\mathbf{q}||^2 = \mathbf{q}^*\mathbf{\Omega q}$ and $\lambda^*||\mathbf{q}||^2 = -\mathbf{q}^*\mathbf{\Omega q}$, we have $(\lambda + \lambda^*)||\mathbf{q}||^2 = 0$.

Since the eigenvector $\mathbf{q}$ cannot be zero, we have $\lambda + \lambda^* = 0$, thus λ is an imaginary number (or zero). □

Lemma 2. *If $\mathbf{\Omega} \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix, we can write $\mathbf{\Omega} = \mathbf{U\Lambda U}^* = \mathbf{U\Lambda U}^{-1}$ where $\mathbf{U} \in \mathbb{C}^{n\times n}$ is a unitary matrix and $\mathbf{\Lambda}$ is a diagonal matrix containing entries that*
(i) are all imaginary (or zero), and
(ii) come in conjugate pairs $-\alpha i, +\alpha i, -\beta i, +\beta i, -\gamma i, +\gamma i \ldots$ $(\alpha, \beta, \gamma \ldots \in \mathbb{R})$.
(If n is odd, it will also have an unpaired zero entry in $\mathbf{\Lambda}$.)

Proof. Because $\mathbf{\Omega}$ is skew-symmetric, $\mathbf{\Omega}^T\mathbf{\Omega} = (-\mathbf{\Omega})(-\mathbf{\Omega}^T) = \mathbf{\Omega\Omega}^T$, i.e. $\mathbf{\Omega}$ is normal. By the spectral theorem $\mathbf{\Omega}$ is diagonalizable by a unitary matrix $\mathbf{U}$ (with $\mathbf{U}^* = \mathbf{U}^{-1}$), i.e.

$$\mathbf{\Omega} = \mathbf{U\Lambda U}^* = \mathbf{U\Lambda U}^{-1}$$

$\mathbf{\Omega}$ and $\mathbf{\Lambda}$ are similar, thus the diagonal entries of $\mathbf{\Lambda}$ are the eigenvalues of $\mathbf{\Omega}$. By Lemma 1, they are all imaginary (or zero). Since $\mathbf{\Omega} \in \mathbb{R}^{n\times n}$, all eigenvalues come in complex conjugate pairs. □

Lemma 3. *If $\mathbf{S} \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, and $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix, then* $\det(\mathbf{S} + \mathbf{A}) \geq \det(\mathbf{S})$.

Proof. Since $\mathbf{S}$ is symmetric positive definite, it can be written in the form $\mathbf{S} = \mathbf{NN}^T$ where $\mathbf{N} \in \mathbb{R}^{n\times n}$ (e.g. from Cholesky factorization). Subsequently, we can write:

$$\begin{aligned}\det(\mathbf{S} + \mathbf{A}) &= \det(\mathbf{NN}^T + \mathbf{A}) \\ &= \det\left[\mathbf{N}(\mathbf{I} + \mathbf{N}^{-1}\mathbf{AN}^{-T})\mathbf{N}^T\right] \\ &= \det(\mathbf{N})\det(\mathbf{I} + \mathbf{N}^{-1}\mathbf{AN}^{-T})\det(\mathbf{N}^T) \\ &= \det(\mathbf{NN}^T)\det(\mathbf{I} + \mathbf{N}^{-1}\mathbf{AN}^{-T}) \\ &= \det(\mathbf{S})\det(\mathbf{I} + \mathbf{\Omega}) \end{aligned} \tag{1}$$

where $\mathbf{\Omega} := \mathbf{N}^{-1}\mathbf{AN}^{-T}$. $\mathbf{\Omega}$ is in fact skew symmetric :

$$\mathbf{\Omega}^T = (\mathbf{N}^{-T})^T\mathbf{A}^T(\mathbf{N}^{-1})^T = -\mathbf{N}^{-1}\mathbf{AN}^{-T} = -\mathbf{\Omega}$$

Thus by Lemma 2 we can write

$$\begin{aligned}\det(\mathbf{I}+\mathbf{\Omega}) &= \det(\mathbf{U}\mathbf{U}^{-1}+\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}) \\ &= \det(\mathbf{U})\det(\mathbf{I}+\mathbf{\Lambda})\det(\mathbf{U}^{-1}) \\ &= \det(\mathbf{I}+\mathbf{\Lambda})\end{aligned}$$

$\mathbf{I}+\mathbf{\Lambda}$ is diagonal with paired imaginary entries $-\alpha i, +\alpha i, -\beta i, +\beta i, -\gamma i, +\gamma i ... (\alpha, \beta, \gamma ... \in \mathbb{R})$. Taking the product of those yields a greater or equal than 1 result since $(1+\alpha i)(1-\alpha i) = 1+\alpha^2 \geq 1$, etc. Hence $\det(\mathbf{I}+\mathbf{\Omega}) \geq 1$. This result, combined with equation 1 yields $\det(\mathbf{S}+\mathbf{A}) \geq \det(\mathbf{S})$.

□

Theorem 4. *Let $\hat{\mathbf{R}} \in \mathbb{R}^{n\times n}$ be a rotation matrix, i.e. $\hat{\mathbf{R}}$ is orthonormal and $\det(\hat{\mathbf{R}}) = 1$, and let $\mathbf{F} \in \mathbb{R}^{n\times n}$. Define $\mathbf{S} = \text{sym}\{\hat{\mathbf{R}}^T\mathbf{F}\}$. If $\mathbf{S} \succ 0$, then $\det(\mathbf{F}) \geq \det(\mathbf{S}) > 0$.*

Proof. The inequality $\det(\mathbf{S}) > 0$ is trivial if $\mathbf{S}$ is positive definite. Since $\hat{\mathbf{R}}$ is a rotation matrix, we have $\det(\mathbf{F}) = \det(\hat{\mathbf{R}})\det(\mathbf{F}) = \det(\hat{\mathbf{R}}^T\mathbf{F})$. Thus if we define $\mathbf{M} = \hat{\mathbf{R}}^T\mathbf{F}$, the theorem becomes equivalent to proving $\det(\mathbf{M}) \geq \det(\mathbf{S})$.

Write $\mathbf{M} = \mathbf{S}+\mathbf{A}$, where $\mathbf{S} = (\mathbf{M}+\mathbf{M}^T)/2$ the symmetric part of $\mathbf{M}$ as previously defined, while $\mathbf{A} = (\mathbf{M}-\mathbf{M}^T)/2$ is the skew-symmetric part of the same matrix. If $\mathbf{S} \succ 0$, then by Lemma 3 we have $\det(\mathbf{M}) = \det(\mathbf{S}+\mathbf{A}) \geq \det(\mathbf{S})$ which completes our proof. □

2 Proof of convexity for our penalty energy term

Finally, we provide a proof for the convexity of the penalty term $E_{penalty}(\mathbf{x}) = \sum_{i,j} p(\lambda_j(\mathbf{S}_i))$ used in our method.

Lemma 5. *For $\forall p : \mathbb{R}^1 \to \mathbb{R}^1$ being a C^1 continuous and convex function, for $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^1$,*

$$(p'(\mathbf{x}_1) - p'(\mathbf{x}_2))(\mathbf{x}_1 - \mathbf{x}_2) \geq 0$$

.

Proof. The follows directly from the fact that the derivative $p'(\mathbf{x})$ is monotonically non-decreasing (due to the convexity of p). □

Lemma 6. *For any square matrices $\mathbf{A}, \mathbf{B}$, and orthogonal matrix $\mathbf{Q}$:*

$$\mathbf{A} : \mathbf{B} = (\mathbf{Q}^T\mathbf{A}\mathbf{Q}) : (\mathbf{Q}^T\mathbf{B}\mathbf{Q})$$

where $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij}b_{ij}$

Proof. Because $\mathbf{Q}$ is orthogonal, $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Thus

$$\begin{aligned}\mathbf{A} : \mathbf{B} &= tr(\mathbf{A}\mathbf{B}^T) \\ &= tr(\mathbf{A}\mathbf{Q}\mathbf{Q}^T\mathbf{B}^T\mathbf{Q}\mathbf{Q}^T) \\ &= tr(\mathbf{Q}^T\mathbf{A}\mathbf{Q} \cdot \mathbf{Q}^T\mathbf{B}^T\mathbf{Q}) && \text{(cyclic permuation invariance of trace)} \\ &= (\mathbf{Q}^T\mathbf{A}\mathbf{Q}) : (\mathbf{Q}^T\mathbf{B}\mathbf{Q})\end{aligned}$$

□

Lemma 7. *For any square matrices $\mathbf{A}, \mathbf{B}$, if A is a diagonal matrix,*

$$\mathbf{A} : \mathbf{B} = \mathbf{A} : diag\{\mathbf{B}\}$$

.

Proof. $\mathbf{A} : \mathbf{B} = \sum_{i=j} a_{ij}b_{ij} + \sum_{i\neq j} a_{ij}b_{ij}$. Because $a_{ij} = 0$ for $i \neq j$, We have

$$\mathbf{A} : \mathbf{B} = \sum_{i=j} a_{ij}b_{ij} = \mathbf{A} : diag\{\mathbf{B}\}$$

□

Theorem 8. *$E_{penalty}(\mathbf{x}) = \sum_{i,j} p(\lambda_j(\mathbf{S}_i))$ is a convex function when p is a C^1 continuous and convex function, where:*
(1) $i = 1, 2, 3...m$, and $j = 1, 2...d$.
(2) m is the number of elements in the mesh, d is the dimension ($d = 2$ for 2D or $d = 3$ for 3D) of the problem.
(3) $\mathbf{S}_i = \text{sym}\{\hat{\mathbf{R}}_i^T \mathbf{F}_i\}$, $\hat{\mathbf{R}}_i$ and $\mathbf{F}_i$ are the ex-rotation field and deformation gradient of the i-th element respectively.
(4) $\lambda_j(\mathbf{S}_i)$ maps from matrix $\mathbf{S}_i$ to its corresponding eigenvalues $\{\lambda_1, \lambda_2...\lambda_d\}$.

Proof. An sufficient condition to prove $E_{penalty}(\mathbf{x})$ being a convex function is that $E_{penalty,i}(\mathbf{x}) = \sum_j p(\lambda_j(\mathbf{S}_i))$ being a convex function for $\forall i$. To make the notation simpler, we will discard the subscript i, and write $\mathbf{S} = \text{sym}\{\hat{\mathbf{R}}^T\mathbf{F}\}$, $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1(\mathbf{S}) & & & \\ & \lambda_2(\mathbf{S}) & & \\ & & ... & \\ & & & \lambda_d(\mathbf{S}) \end{bmatrix}$.

Notice that now we want to prove $E_{penalty,i} = \varphi(\mathbf{\Lambda}(\mathbf{S}(\mathbf{x}))) = \sum_j p(\lambda_j(\mathbf{S}))$ is a convex function over $\mathbf{x}$. Because $\mathbf{S}$ is a linear mapping of $\mathbf{x}$, it is sufficient to just prove $\varphi(\mathbf{\Lambda}(\mathbf{S}))$ is convex over S, so problem turns to be :

$$\delta\mathbf{S} : \frac{\partial^2\varphi(\mathbf{\Lambda}(\mathbf{S}))}{\partial\mathbf{S}^2} : \delta\mathbf{S} \geq 0$$

or

$$\delta_{\mathbf{S}}(\frac{\partial\varphi(\mathbf{\Lambda}(\mathbf{S}))}{\partial\mathbf{S}}) : \delta\mathbf{S} \geq 0$$

Let's take a look at $\frac{\partial\varphi(\mathbf{\Lambda}(\mathbf{S}))}{\partial\mathbf{S}}$ first :

$$\delta_{\mathbf{S}}\varphi(\mathbf{\Lambda}) = \nabla\varphi(\mathbf{\Lambda}) : \delta_{\mathbf{S}}(\mathbf{\Lambda}) \qquad \nabla\varphi(\mathbf{\Lambda}) = \begin{bmatrix} p'(\lambda_1) & & \\ & ... & \\ & & p'(\lambda_d) \end{bmatrix}$$

Since $\mathbf{\Lambda}$ comes from an eigen decomposition from $\mathbf{S}$, $\mathbf{Q\Lambda Q}^T = \mathbf{S}$, we have

$$\begin{aligned} \delta_{\mathbf{S}}\mathbf{Q\Lambda Q}^T + \mathbf{Q}\delta_{\mathbf{S}}\mathbf{\Lambda Q}^T + \mathbf{Q\Lambda}\delta_{\mathbf{S}}\mathbf{Q}^T &= \delta\mathbf{S} \\ Q^T(\delta_{\mathbf{S}}Q\mathbf{\Lambda Q}^T + \mathbf{Q}\delta_{\mathbf{S}}\mathbf{\Lambda Q}^T + \mathbf{Q\Lambda}\delta_{\mathbf{S}}\mathbf{Q}^T)\mathbf{Q} &= \mathbf{Q}^T\delta\mathbf{SQ} \\ (\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})\mathbf{\Lambda} + \delta_{\mathbf{S}}\mathbf{\Lambda} + \mathbf{\Lambda}(\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})^T &= \mathbf{Q}^T\delta\mathbf{SQ} \end{aligned}$$

Notice that $\mathbf{QQ}^T = \mathbf{I}$,

$$(\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})^T + \mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q} = 0$$

Thus, $\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q}$ is a skew-symmetric matrix, and $(\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})\mathbf{\Lambda}+\mathbf{\Lambda}(\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})^T$ would be an off-diagonal matrix. Hence $\delta_{\mathbf{S}}\mathbf{\Lambda} = diag\{\mathbf{Q}^T\delta\mathbf{SQ}\}$. Therefore,

$$\begin{aligned} \delta_{\mathbf{S}}(\varphi(\mathbf{\Lambda}(\mathbf{S}))) &= \nabla\varphi(\mathbf{\Lambda}) : \delta_{\mathbf{S}}\mathbf{\Lambda} \\ &= \nabla\varphi(\mathbf{\Lambda}) : diag\{\mathbf{Q}^T\delta\mathbf{SQ}\} \\ &= \nabla\varphi(\mathbf{\Lambda}) : \mathbf{Q}^T\delta\mathbf{SQ} && (Lemma\ 7) \\ &= \mathbf{Q}\nabla\varphi(\mathbf{\Lambda})\mathbf{Q}^T : \delta\mathbf{S} && (Lemma\ 6) \end{aligned}$$

That's to say : $\frac{\partial\varphi(\mathbf{\Lambda}(\mathbf{S}))}{\partial\mathbf{S}} = \mathbf{Q}\nabla\varphi(\mathbf{\Lambda})\mathbf{Q}^T$ by definition. Now let's prove $\delta_{\mathbf{S}}(\frac{\partial\varphi(\mathbf{\Lambda}(\mathbf{S}))}{\partial\mathbf{S}}) : \delta\mathbf{S} \geq 0$:

$$\begin{aligned}\delta_{\mathbf{S}}(\frac{\partial\varphi(\mathbf{\Lambda}(\mathbf{S}))}{\partial\mathbf{S}}):\delta\mathbf{S} &= \delta_{\mathbf{S}}(\mathbf{Q}\nabla\varphi(\mathbf{\Lambda})\mathbf{Q}^T):\delta\mathbf{S} \\ &= \delta_{\mathbf{S}}(\mathbf{Q}\nabla\varphi(\mathbf{\Lambda})\mathbf{Q}^T):\delta_{\mathbf{S}}(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T) \\ &= (\mathbf{Q}^T\delta_{\mathbf{S}}(\mathbf{Q}\nabla\varphi(\mathbf{\Lambda})\mathbf{Q}^T)\mathbf{Q}):(\mathbf{Q}^T\delta_{\mathbf{S}}(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T)\mathbf{Q}) \qquad (Lemma\ 6) \\ &= ((\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})\nabla\varphi(\mathbf{\Lambda})+\delta_{\mathbf{S}}(\nabla\varphi(\mathbf{\Lambda}))+\nabla\varphi(\mathbf{\Lambda})(\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})^T) \\ &\quad :((\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})\mathbf{\Lambda}+\delta_{\mathbf{S}}\mathbf{\Lambda}+\mathbf{\Lambda}(\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})^T)\end{aligned}$$

Notice that $\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q}$ is a skew-symmetric matrix, we can group the diagonal terms and off-diagonal terms separately, thus

$$\delta_{\mathbf{S}}(\frac{\partial\varphi(\mathbf{\Lambda}(\mathbf{S}))}{\partial\mathbf{S}}):\delta\mathbf{S} = \underbrace{((\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})\nabla\varphi(\mathbf{\Lambda})+\nabla\varphi(\mathbf{\Lambda})(\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})^T):((\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})\mathbf{\Lambda}+\mathbf{\Lambda}(\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q})^T)}_{(*)} + \underbrace{\delta_{\mathbf{S}}(\nabla\varphi(\mathbf{\Lambda})):\delta_{\mathbf{S}}\mathbf{\Lambda}}_{(**)}$$

If we write down the skew-symmetric matrix $\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q}$ explicitly as

$$\mathbf{Q}^T\delta_{\mathbf{S}}\mathbf{Q} = \begin{bmatrix} 0 & q_{12} & & & q_{1d} \\ -q_{12} & 0 & & & . \\ . & & . & & . \\ . & & & 0 & q_{d-1,d} \\ -q_{1d} & . & . & -q_{d-1,d} & 0 \end{bmatrix},$$

we can expand $(*)$ to

$$\begin{aligned}(*) &= \begin{bmatrix} 0 & ((p'(\lambda_2))-p'(\lambda_1))q_{12} & & & ((p'(\lambda_d))-p'(\lambda_1))q_{1d} \\ ((p'(\lambda_2))-p'(\lambda_1))q_{12} & 0 & & & . \\ . & & . & & . \\ . & & & 0 & ((p'(\lambda_d))-p'(\lambda_{d-1}))q_{d-1,d} \\ ((p'(\lambda_d))-p'(\lambda_1))q_{1d} & . & . & ((p'(\lambda_d))-p'(\lambda_{d-1}))q_{d-1,d} & 0 \end{bmatrix} \\ &\quad : \begin{bmatrix} 0 & (\lambda_2-\lambda_1)q_{12} & & & (\lambda_d-\lambda_1)q_{1d} \\ (\lambda_2-\lambda_1)q_{12} & 0 & & & . \\ . & & . & & . \\ . & & & 0 & (\lambda_d-\lambda_{d-1})q_{d-1,d} \\ (\lambda_d-\lambda_1)q_{1d} & . & . & (\lambda_d-\lambda_{d-1})q_{d-1,d} & 0 \end{bmatrix} \\ &= 2\sum_{k<l}(p'(\lambda_l)-p'(\lambda_k))(\lambda_l-\lambda_k)q_{kl}^2\end{aligned}$$

Since function p is C^1 continuous and convex, we have $(p'(\lambda_l)-p'(\lambda_k))(\lambda_l-\lambda_k)\geq 0$ by applying Lemma 5, thus $(*)\geq 0$.

Similarly, we can expand $(**)$ to

$$(**) = \sum_{k=1}^{d} p''(\lambda_k)(\delta_S(\lambda_k))^2$$

Once again because p is a convex function, $p''(\lambda_k)\geq 0$. Thus $(**)\geq 0$.

Therefore, we proved that $\delta_{\mathbf{S}}(\frac{\partial\varphi(\mathbf{\Lambda}(\mathbf{S}))}{\partial\mathbf{S}}):\delta\mathbf{S}\geq 0$, and $E_{penalty}(\mathbf{x})=\sum_{i,j}p(\lambda_j(\mathbf{S}_i))$ is a convex function. □